Lecture 16. Morrey's Inequality.

Morrey's Inequality. *If $p > n$ then there exists $C = C(p, n)$ such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \;\leq\; C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$ where

$$\gamma := 1 - n/p.$$

In fact we obtain the more refined estimate that if $|x - y| = r$ then

$$|u(x)| \;\leq\; C(r)\|u\|_{W^{1,p}(B(x,r))}, \qquad |u(x) - u(y)| \;\leq\; Cr^{1-n/p}\|Du\|_{L^p(B(x,2r))}.$$

Notation

$$⨍_W u\,dx \;=\; \frac{1}{\text{vol }(W)}\int_W u\,dx.$$

Last time:

$$\int_{B(x,r)} |u(y) - u(x)|\,dy \;\leq\; C(r)\int_{B(x,r)} \frac{|Du(y)|}{|y - x|^{n-1}}\,dy.$$

In fact, we calculated

$$C(r) \;=\; \frac{r^n}{n}.$$

Hence

$$⨍_{B(x,r)} |u(y) - u(x)|\,dy \;\leq\; C\int_{B(x,r)} \frac{|Du(y)|}{|y - x|^{n-1}}\,dy.$$

Continuing from there, for $x, y \in \mathbb{R}^n$ with $|x - y| = r$, take $W = B(x, r) \cap B(y, r)$. For $w \in W$ we have

$$|u(y) - u(x)| \;\leq\; |u(y) - u(w)| \;+\; |u(x) - u(w)|,$$

and averaging over W, we get

$$|u(x) - u(y)| \;\leq ⨍_W |u(x) - u(w)|\,dw \;+ ⨍_W |u(w) - u(y)|\,dw.$$

But since the volume of W is a fixed (independent of r) multiple of the volume of $B(x, r)$, we have

(*)

$$⨍_W |u(x) - u(w)|\,dw \;\leq\; C ⨍_{B(x,r)} |u(x) - u(w)|\,dw \;\leq\; C'\int_{B(x,r)} \frac{|Du(w)|}{|w - x|^{n-1}}\,dw$$

$$\leq\; C'\left(\int_{B(x,r)} |Du|^p\,dw\right)^{1/p}\left(\int_{B(x,r)} \frac{dw}{|x - w|^{(n-1)p/(p-1)}}\right)^{(p-1)/p}.$$

However, the second integral on the right is finite provided

$$\frac{(n-1)p}{p-1} < n \Leftrightarrow \frac{p}{p-1} < \frac{n}{n-1} \Leftrightarrow n < p.$$

We use polar coordinates to see that

$$\begin{aligned}\left(\int_{B(x,r)} \frac{dw}{|x-w|^{(n-1)p/(p-1)}}\right)^{(p-1)/p} &= \left(|\partial B(0,1)| \int_0^r s^{n-1-(n-1)p/(p-1)}\, ds\right)^{(p-1)/p} \\ &= c(n,p)\left(r^{(p-n)/(p-1)}\right)^{(p-1)/p} = c(n,p) r^{1-n/p}.\end{aligned}$$

In fact, putting this together and using the fact that $B(y,r) \subset B(x,2r)$, gives

$$|u(x)-u(y)| \le C(n,p) r^{1-n/p} \left(\int_{B(x,2r)} |Du|^p\, dw\right)^{1/p}.$$

On the other hand,

$$\begin{aligned}(**) \quad |u(x)| &\le ⨍_{B(x,1)} |u(x)-u(y)|\, dy + ⨍_{B(x,1)} |u(y)|\, dy \\ &\le C\left(\int_{B(x,1)} |Du|^p\, dw\right)^{1/p} + C\left(\int_{B(x,1)} |u(y)|^p\, dy\right)^{1/p}.\end{aligned}$$

This proves Morrey's inequality.

Corollary 1. *If $u \in W^{1,p}(\mathbb{R}^n)$ then*

(a). If $p < n$ then $u \in L^{p^*}(\mathbb{R}^n)$ and there exists $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

(b). If $p > n$ then u has a continuous version, meaning there exists a continuous function u^* with $u = u^*$ almost everywhere, and for the continuous version we have $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma = 1 - n/p$. Moreover, there exists $C = C(p,n)$ such that

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Proof. (a) followed from G-N-S, and (b) follows from Morrey's theorem by approximating u by functions in C_0^∞ and taking the limit. Indeed, choose $u_j \to u$ in $W^{1,p}(\mathbb{R}^n)$ with $u_j \in C_c^\infty(\mathbb{R}^n)$. Then by Morrey's inequality, u_j is Cauchy in $C^{0,\gamma}(\mathbb{R}^n)$, and hence tends to a limit $u^* \in C^{0,\gamma}(\mathbb{R}^n)$. But then $u = u^*$ almost everywhere, and

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} = \lim_{j\to\infty} \|u_j\|_{C^{0,\gamma}(\mathbb{R}^n)} \le \lim_{j\to\infty} C\|u_j\|_{W^{1,p}(\mathbb{R}^n)} = C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Corollary 2. *Corollary 1 holds when $\mathbb{R}^n$ is replaced by an arbitrary bounded open set $U \subset \mathbb{R}^n$ with C^1 boundary.*

Proof. Part (a) was dealt with before. For (b), we introduce an extension operator then make use of the result on $\mathbb{R}^n$. Indeed, we have a bounded operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$, with $Eu|_U = u$. Then Eu has a continuous version $(Eu)^*$. Let $u^* = (Eu)^*_U$. Then

$$\|u^*\|_{C^{0,\gamma}(U)} \;\leq\; C\|(Eu)^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \;\leq\; C\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \;\leq\; C\|u\|_{W^{1,p}(U)}.$$

It is convenient to make the convention that we always work with the continuous version of a function in $W^{1,p}(U)$ when one exists.

Define $p^{(r)}$ by

$$\frac{1}{p^{(r)}} = \frac{1}{p} - \frac{r}{n},$$

so $(p^{(r)})^* = p^{(r+1)}$. Also, for $s \in \mathbb{R}$, we define

$$\begin{aligned} [s] &= \text{ the largest integer } \leq s, \\ \{s\} &= s - [s]. \end{aligned}$$

Theorem. *Let U be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Assume $u \in W^{k,p}(U)$. There exists a constant $C = C(n,k,p,U)$ such that*
(a). If $r < n/p$ and $r \leq k$ then $u \in W^{k-r,p^{(r)}}(U)$, and

$$\|u\|_{W^{k-r,p^{(r)}}(U)} \;\leq\; C\|u\|_{W^{k,p}(U)}.$$

(b). If $k > n/p$ and n/p is not an integer then $u \in C^{[k-n/p],\{k-n/p\}}(U)$ and

$$\|u\|_{C^{0,\gamma}(U)} \;\leq\; C\|u\|_{W^{k,p}(U)}.$$

Remark. Regarding (a), let us think about how the Sobolev indices $(k-r, p^{(r)})$ relate to the indices (k,p). what we are doing is using up r of the k derivatives available to increase the exponent p to $p^{(r)}$. We can explain qualitatively the formula

$$\frac{1}{p^{(r)}} \;=\; \frac{1}{p} \;-\; \frac{r}{n}.$$

The negative sign on the right is there because when we trade in derivatives, we increase p. The term r/n on the right occurs because the more derivatives we trade in, the more gain in p we get, however, the higher the dimension, the less effect each derivative has. For (b), in going from the Sobolev space $W^{k,p}$ to a C^k space, one looses n/p derivatives.

Proof of the Theorem. (a). Suppose $u \in W^{k,p}(U)$. Then when $|\alpha| \le k-1$ we have $D^\alpha u \in W^{1,p}(U)$. Hence by Corollary 2, $D^\alpha u \in L^{p^*}(U) = L^{p^{(1)}}(U)$. Hence $u \in W^{k-1,p^{(1)}}(U)$. Then by induction we see that

$$u \in W^{k,p}(U) \ \subset \ W^{k-1,p^{(1)}}(U) \subset W^{k-2,p^{(2)}}(U) \subset \ \ldots \ \subset W^{k-r,p^{(r)}}(U).$$

The only condition that we need for the induction to proceed is $r \le k$ and $p^{(j)} < n$ for $j < r$. However,

$$p < p^{(1)} < p^{(2)} < \cdots < p^{(r-1)},$$

and

$$p^{(r-1)} < n \ \Leftrightarrow \ \frac{1}{p^{(r-1)}} > \frac{1}{n} \ \Leftrightarrow \ \frac{1}{p} - \frac{r-1}{n} > \frac{1}{n} \ \Leftrightarrow \ \frac{1}{p} > \frac{r}{n} \ \Leftrightarrow \ r < \frac{n}{p}.$$

(b). Suppose $u \in W^{k,p}(U)$ and start by assuming that $n < p$. For $|\alpha| \le k-1$ we have $D^\alpha u \in W^{1,p}(U)$ and so by Corollary 2, $D^\alpha u \in C^{0,1-n/p}(U)$. But then $u \in C^{k-1,1-n/p}(U)$. However, since $0 < n/p < 1$, we have

$$C^{k-1,1-n/p}(U) = C^{[k-n/p],\{k-n/p\}}(U).$$

Now suppose $p < n$, and choose ℓ so that

$$p < p^{(1)} < \cdots < p^{(\ell-1)} < n < p^{(\ell)}.$$

Then by part (a), since $u \in W^{k,p}(U)$ we have $u \in W^{k-\ell,p^{(\ell)}}(U)$, but then since $p^{(\ell)} > n$ we get

$$u \in C^{[k-\ell-n/p^{(\ell)}],\{n-\ell-n/p^{(\ell)}\}}(U).$$

However,

$$k - \ell - \frac{n}{p^{(\ell)}} \ = \ k - \ell - n\left(\frac{1}{p} - \frac{\ell}{n}\right) \ = \ k - \frac{n}{p}.$$

This completes the proof.