MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 17. Compactness.

Definition Suppose that $T : X \to Y$ is a bounded linear map between two Banach spaces. We say that T is *compact* if the set $T(\{x \in X : \|x\| \leq 1\})$ is precompact in Y. Equivalently, if whenever x_n is a bounded sequence in X, then there exists a subsequence x_{n_j}, and an element $y \in Y$ with $Tx_{n_j} \to y$ in Y.

Arzela-Ascoli compactness criterion for uniform convergence. *Let K be a compact subset of $\mathbb{R}^n$ and suppose that Ω is a subset of $C(K)$ which is bounded and equicontinuous, meaning that there exists M such that*

$$|f(x)| \ \leq \ M \qquad \text{for all } f \in \Omega,\ x \in K,$$

and given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \qquad \text{for all } f \in \Omega,\ x, y \in K.$$

Then Ω is precompact in $C(K)$.

Theorem 1. *Let U be an open subset of $\mathbb{R}^n$. A bounded set $\Omega \subset L^p(U)$ is precompact in $L^p(U)$ if an only if for every $\varepsilon > 0$, there exists $\delta > 0$ and a compact set $K \subset U$ such that for every $u \in \Omega$,*

$$\int_{U \setminus K} |u|^p \, dx \ < \ \varepsilon,$$

and

$$\int_K |u(x+y) - u(x)|^p \, dx < \varepsilon \quad \text{when } |y| \leq \delta.$$

Theorem 1′. *Let K be a compact subset of $\mathbb{R}^n$. Suppose $\Omega \subset L^p(\mathbb{R}^n)$ is bounded in $L^p(\mathbb{R}^n)$ and every $f \in \Omega$ is supported in K. Then Ω is precompact in $L^p(\mathbb{R}^n)$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $u \in \Omega$*

$$(*) \qquad \int_{\mathbb{R}^n} |u(x+y) - u(x)|^p \, dx \ < \ \varepsilon \quad \text{when } |y| \leq \delta.$$

Proof of Theorem 1′. We will only need the "if" part of the Theorem, so the other part is an exercise. For $\delta > 0$, define

$$\eta_\delta(x) \ = \ \frac{1}{\delta^n} \eta\left(\frac{x}{\eta}\right),$$

where η is a non-negative function supported in $B(0,1)$ with $\int \eta = 1$ and $\eta(-y) = \eta(y)$ for all y. For $u \in \Omega$, we have $\eta_\delta * u$ is smooth and supported in the compact set $K_\delta = \{x : \mathrm{dist}(x, K) \leq \delta\}$. For fixed $\delta > 0$, consider the set

$$\Omega_\delta = \{\eta_\delta * u : u \in \Omega\}.$$

Now Ω_δ is bounded and equicontinuous and is thus precompact in $C(K_\delta)$ and hence also in $L^p(\mathbb{R}^n)$. Indeed,

$$\eta_\delta * u(x) \;=\; \int_{\mathbb{R}^n} \eta_\delta(y)u(x+y)\,dy,$$

So by Hölder's inequality,

$$|\eta_\delta * u(x)| \;\le\; C\|u\|_{L^p(\mathbb{R}^n)}, \qquad\qquad C = C(\delta) = \left(\int_{\mathbb{R}^n} |\eta_\delta|^{p'}\right)^{1/p'}.$$

Similarly since $D\eta_\delta$ is bounded in $L^{p'}$, we have a constant C independent of $u \in L^p(\mathbb{R}^n)$ (but depending on δ), such that

$$|D(\eta_\delta * u)(x)| \;=\; |((D\eta_\delta) * u)(x)| \;\le\; C\|u\|_{L^p}.$$

Thus Ω_δ is bounded and equicontinuous.

Now

$$(\eta_\delta * u \;-\; u)(x) \;=\; \int_{\mathbb{R}^n} \eta_\delta(y)((u(x+y) - u(x))\,dy,$$

so by Jensen's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} |(\eta_\delta * u \;-\; u)(x)|^p\,dx \;&\le\; \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \eta_\delta(y)|u(x+y) - u(y)|^p\,dydx \\ &\le\; \sup_{|y|<\delta} \int_{\mathbb{R}^n} |u(x+y) - u(y)|^p\,dy.\end{aligned}$$

By (*), we see that

$$(**) \qquad \sup_{u\in\Omega} \|\eta_\delta * u - u\|_{L^p(\mathbb{R}^n)} \;\le\; \varepsilon(\delta) \;\to\; 0 \quad \text{as } \delta \to 0.$$

Now given a sequence u_j in Ω, we make a diagonal argument to get a convergent subsequence. Indeed, we can take a subsequence $u_j^{(1)}$ so that $\eta_1 * u_j^{(1)}$ converges to $u^{(1)}$ in $L^p(\mathbb{R}^n)$. Take a subsequence of this $u_j^{(2)}$ so that $\eta_{1/2} * u_j^{(2)}$ converges to some function $u^{(2)}$ in $L^p(\mathbb{R}^n)$. Repeat this procedure to get a subsequence $u_j^{(k+1)}$ of $u_j^{(k)}$ such that $\eta_{1/k} * u_j^{(k)}$ converges to $u^{(k)}$ in $L^p(\mathbb{R}^n)$. Now take the diagonal subsequence $\tilde{u}_j = u_j^{(j)}$, which is (apart from the first k values) a subsequence of the sequence $u_j^{(k)}$. then

$$\eta_{1/k} * \tilde{u}_j \;\to\; u^{(k)} \text{ as } j \to \infty,$$

for all $k = 1, 2, \ldots$. However, by (**),

$$\|\eta_{1/k} * \tilde{u}_j - \tilde{u}_j\|_{L^p} \;\le\; \varepsilon(1/k)C \;\to\; 0 \quad \text{as } k \to \infty$$

uniformly in j. Hence $\tilde{u}_j$ is Cauchy in $L^p(\mathbb{R}^n)$. Indeed,

$$\|\tilde{u}_j - \tilde{u}_\ell\| \;\le\; \|\tilde{u}_j - \delta_{1/k} * \tilde{u}_j\| +$$

$\square$

Rellich-Kondrachov Compactness Theorem. *Let U be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. If $1 \leq p < \infty$, then the inclusion of $W^{1,p}(U)$ into $L^q(U)$ is compact if $1 \leq q < p^*$.*

Exercise. Show that if U is a bounded open subset of $\mathbb{R}^n$ and if $1 \leq p < q < \infty$ then the inclusion $L^q(U) \to L^p(U)$ is not compact.

Proof of Rellich-Kondrachov. Take a bounded extension operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ with Eu supported in $\overline{B(0,M)}$ for some $M > 0$. Then for some $m > 0$ we have
(***)
$$\{Eu : \|u\|_{W^{1,p}(U)} \leq 1\} \;\subset\; \{u \in W^{1,p}(\mathbb{R}^n) : \mathrm{supp}(u) \subset \overline{B(0,M)},\; \|u\|_{W^{1,p}(U)} \leq m\}.$$

The set on the right is bounded in $L^{p^*}(\mathbb{R}^n)$ by the Sobolev inequality, and hence by Hölder's inequality it is bounded in $L^q(U)$. However, considering now $u \in C^1_c(\mathbb{R}^n)$ supported in $\overline{B(0,M)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x+y) - u(x)|\, dx \;&\leq\; \int_{\mathbb{R}^n} \int_0^1 \left| \frac{d}{dt} u(x+ty) \right| dtdy \\ &\leq\; |y| \int_0^1 \int_{\mathbb{R}^n} |Du(x+ty)|\, dtdx \;\leq\; |y| \int_{\mathbb{R}^n} |Du|\, dx \;\leq\; C|y| \|Du\|_{L^{p^*}(\mathbb{R}^n)}.\end{aligned}$$

Now applying Hölder's inequality to $f^r f^{q-r}$ with exponents $1/r$ and $1/(1-r)$ we get

$$\int f^q \;\leq\; \left(\int f\right)^r \left(\int f^{(q-r)/(1-r)}\right)^{1-r} \;=\; \left(\int f\right)^{(p^*-q)/(p^*-1)} \left(\int f^{p^*}\right)^{(q-1)/(p^*-1)}.$$

if we choose $(q-r)/(1-r) = p^*$ by taking $r = (p^* - q)/(p^* - 1)$. Then we apply this to the function $f(x) = u(x+y) - u(x)$ to see that if $\|u\|_{W^{1,p}} \leq m$ then

$$\int_{\mathbb{R}^n} |u(x+y) - u(x)|^q\, dx \;\leq\; C \left(\int_{\mathbb{R}^n} |u(x+y) - u(x)|\, dx \right)^{(p^*-q)/(p^*-1)} \;\leq\; C'|y|^{(p^*-q)/(p^*-1)},$$

where $C' = C'(M,m)$ is independent of u and $|y|$. Hence applying this with u replaced by Eu from above, we get (*) satisfied for the exponent q where δ is proportional to $\varepsilon^{(p^*-1)/(p^*-q)}$. Hence the sets in (***) are precompact in $L^q(\mathbb{R}^n)$ and so if u_j is a bounded sequence in $W^{1,p}(U)$ then u_j has a subsequence u_{j_n} such that Eu_{j_n} converges in $L^q(\mathbb{R}^n)$. However, this implies u_{j_n} converges in $L^q(U)$.