Lecture 18. Odds and Ends: Poincaré, Difference quotients, H^{-1}.

See Sections 5.8 and 5.9 of the text.

Poincaré Inequality. *Let U be a bounded, connected, open subset of $\mathbb{R}^n$ with C^1 boundary ∂U. Assume $1 \le p \le \infty$. Then there exists a constant C depending only on n, p, U such that*

$$\|u - \bar{u}\|_{L^p(U)} \ \le \ C\|Du\|_{L^p(U)}, \qquad\qquad \bar{u} = ⨍_U u.$$

Corollary. *There exists a constant $C' = C'(n, p, U)$ such that*

$$\|u\|_{W^{1,p}(U)} \ \le \ C' \left(\|Du\|_{L^p(U)} \ + \ |\bar{u}|\right).$$

Definition. The space H^{-1} is the dual space of $H_0^1(U)$, that is the space of bounded linear functionals on $H_0^1(U)$ equipped with the norm

$$\|f\|_{H^{-1}(U)} \ = \ \sup\{f(u) : u \in H_0^1(U), \ \|u\|_{H_0^1(U)} \le 1\}.$$

Theorem. *(i) Assume $f \in H^{-1}(U)$. Then there exist functions $f^1, f^1, \ldots, f^n$ in $L^2(U)$ such that*

$$(1) \qquad f(v) \ = \ \int_U \left(f^0 u \ + \sum_{i=1}^n f^i v_{x_i}\right) dx \qquad \textit{for all } v \in H_0^1(U).$$

(ii) Furthermore,

$$\|H^{-1}(U)\| \ = \ \inf\left\{\left(\int_U \sum_{i=0}^n |f^1|^2 \, dx\right)^{1/2} \ : \ f(u) \textit{ is given by (1) for } f^1, \ldots, f^n \in L^2(U)\right\}.$$

Notation. $f = f^0 - \sum_{i=1}^n f^i_{x_i}$.

Remark. For $f \in L^1(\mathbb{R}^n)$ define the Fourier transform

$$\hat{f}(\xi) \ = \ \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx.$$

Then $f \to \hat{f}$ extends to an isometry of $L^2(\mathbb{R}^n)$. We have

$$\hat{\hat{f}}(x) \ = \ f(-x),$$

so

$$f(x) \; = \; \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} \, dx.$$

The Fourier transform diagonalizes differentiation. Indeed,

$$\widehat{D^\alpha f}(\xi) \; = \; (i\xi)^\alpha \hat{f}(\xi).$$

Using this, we have

$$\|f\|_{H^k(\mathbb{R}^n)} \; \sim \; \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.$$

This enables us to extend the definition of $H^k(\mathbb{R}^n)$ to $k \in \mathbb{R}$.

Definition. The ith -difference quotient of size h is

$$D_i^h u(x) \; = \; \frac{u(x+he_i)-u(x)}{h}, \qquad\qquad i=1,\ldots,n.$$

Then $D^h u := (D_1^h u, \ldots, D_n^h u)$.

Lemma. *(Evans, Theorem 3, Section 5.8.2) (a). For* $1 \le p < \infty$ *and* $0 < |h| <$ *dist*$(V, \partial U)$ *there exists* $C = C(V,U)$ *such that*

$$\|D^h u\|_{L^p(V)} \; \le \; C\|Du\|_{L^p(U)}, \qquad\qquad \text{for all } u \in W^{1,p}(U).$$

(b). Conversely, if $1 < p < \infty$ *and* $u \in L^p(U)$*, and there exists* C *and* ε *with* $0 < \varepsilon <$ *dist*$(V, \partial U)$ *such that*

$$\|D^h u\|_{L^p(V)} \; \le \; C, \qquad\qquad \text{for all } 0 < |h| < \varepsilon,$$

then $u \in W^{1,p}(U)$ *and*

$$\|Du\|_{L^p(V)} \; \le \; C.$$