Lecture 19. Elliptic Equations. Consider the equation

$$(*) \qquad \begin{cases} Lu = f & \text{in } u \\ u = 0 & \text{on } \partial U. \end{cases}$$

Here, U is open and bounded. We are looking for $u : \bar{U} \to \mathbb{R}$ and

$$(**) \qquad Lu \;=\; -\sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_i}\right)_{x_j} \;+\; \sum_{i=1}^{n} b^i(x) u_{x_i} \;+\; c(x)u$$

is a linear second order operator written in *divergence form*. We assume that a^{ij}, b^i and c are in $L^\infty(U)$ and L is *elliptic*, meaning that there is a constant θ such that

$$\sum_{i,j} a^{ij}(x)\xi_i\xi_j \;\geq\; \theta|\xi|^2,$$

for almost every $x \in U$ and all $\xi \in \mathbb{R}^n$.

Weak Solution. If we have a classical solution of (*) then for any test function $v \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} \;+\; \sum_{i=1}^{n} b^i u_{x_i} v \;+\; cuv \;=\; \int_U fv\,dx.$$

Definition. The bilinear form $B[\ ,\ ]$ associated with the divergence form ellliptic operator L is

$$B[u,v] \;=\; \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} \;+\; \sum_{i=1}^{n} b^i u_{x_i} v \;+\; cuv.$$

We say that $u \in H_0^1(U)$ is a *weak solution* of the equation (*) if

$$B[u,v] \;=\; \int fv\,dx$$

for all $v \in H_0^1(U)$, where $(u,v) = \int_U uv\,dx$ is the inner product on $L^2(U)$.

Riesz Representation Theorem. *If $(H, \langle\ ,\ \rangle)$ is a Hilbert space and ℓ is a bounded linear functional on H, then there exists a unique element u of H such that for all $v \in H$,*

$$\ell(v) \;=\; \langle u, v\rangle.$$

Theorem. *If $b_i = 0$ for all i and $c \geq 0$, then there exists a weak solution to (*) for every $f \in L^2(U)$.*

Proof. Since $(b_1, \ldots, b_n) = 0$, the bilinear form $B[u,v]$ is symmetric. Now there exists $\varepsilon > 0$ such that

$$\varepsilon \|u\|^2_{H^1_0(U)} \;\leq\; B[u,u]. \tag{3}$$

Indeed,

$$\int \sum_{i,j} a_{ij} u_{x_i} u_{x_j}\, dx \;\geq\; \theta \int |Du|^2\, dx,$$

so

$$B[u,u] \;\geq\; \theta \|Du\|^2_{L^2(U)} \;+\; \int cu^2\, dx \;\geq\; \theta \|Du\|^2_{L^2(U)}.$$

If c is bounded below by a positive constant on U we are done, but in any case we showed that there exists C such that

$$C\|Du\|_{L^2(U)} \;\geq\; \|u\|_{H^1_0(U)},$$

so we get (3). On the other hand, we can see that there exists C such that

$$B[u,v] \;\leq\; C\|u\|_{H^1_0(U)}\|v\|_{H^1_0(U)}.$$

Indeed,

$$\left| \int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j}\, dx \right| \;\leq\; \left\| \left(\sum_{i,j} (a^{ij})^2 \right)^{1/2} \right\|_{L^\infty(U)} \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)}. \tag{4}$$

Hence the norm given by the inner product B is comparable to the standard norm on $H^1_0(U)$, and $(H^1_0(U), B)$ forms a Hilbert space.

Now

$$\int fv\, dx \;\leq\; \|f\|_{L^2(U)}\|v\|_{L^2(U)} \;\leq\; C\|f\|_{L^2(U)} B[u,u]^{1/2}.$$

Hence by the RRT there exists $u \in H^1_0(U)$ such that

$$B[u,v] = \int fv\, dx \qquad \text{for all } v \in H^1_0(U).$$

In the case $b = 0$ and $c \geq 0$, the bilinear form B is an inner product comparable to the standard inner product on $H^1_0(U)$. Indeed, for some $C > 0$ we have

$$B[u,v] \;\leq\; C\|u\|_{H^1(U)}\|v\|_{H^1(U)},$$

and the ellipticity condition ensures that

$$\varepsilon\|Du\|_{L^2(U)} \;\leq\; B[u,u].$$

In fact we have that on $H_0^1(U)$

$$\|u\|_{H^1(U)} \;\leq\; C\|Du\|_{L^2(U)},$$

so

$$\varepsilon\|u\|_{H^1(U)} \;\leq\; B[u,u].$$

Hence there exists a unique $u \in H_0^1(U)$ such that for every $v \in H_0^1(U)$ we have

$$B[u,v] \;=\; (f,v).$$

In other words, there is a unique weak solution $u \in H_0^1(U)$ to (*).

Theorem. *There exists $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of the boundary value problem*

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

We will use

Lax-Milgram Theorem. *Let $(H, \langle\ ,\ \rangle)$ be a Hilbert space. Assume that*

$$B : H \times H \to \mathbb{R}$$

is a bilinear mapping such that there exist constants $\alpha, \beta > 0$ with

$$|B[u,v]| \;\leq\; \alpha\|u\|\|v\|, \qquad\qquad u,v \in H$$

and

$$\beta\|u\|^2 \;\leq\; B[u,u] \qquad\qquad u \in H.$$

Then if f is a continuous linear functional on H, there exists a unqiue element $u \in H$ such that

$$B[u,v] = f(v), \qquad\qquad \text{for all } v \in H.$$

Proof of the existence theorem. Energy Estimates: Set

$$B_\gamma[u,v] \;=\; B[u,v] \;+\; \gamma(u,v)$$

where B is the bilinear form associated to L. Then clearly there exists $C_\gamma < \infty$ such that

$$B[u,v] \;\leq\; C_\gamma\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)}.$$

Indeed, the first term is bounded in (4). For the other terms we have

$$\left|\int_U \sum_i b^i u_{x_i} v\, dx\right| \;\leq\; \int_U \left(\sum_i (b^i)^2\right)^{1/2} |Du||v|\, dx \;\leq\; \left\|(\sum_i (b^i)^2)^{1/2}\right\|_{L^\infty(U)} \|Du\|_{L^2}\|v\|_{L^2(U)},$$

and

$$\left|\int_U cuv\, dx\right| \;\leq\; \|c\|_{L^\infty(U)}\|u\|_{L^2(U)}\|v\|_{L^2(U)}.$$

From which we get the result. However, we also have a lower bound for $B[u,u]$. Indeed, using the fact that for $\varepsilon > 0$,

$$\alpha\beta \;\leq\; \frac{\varepsilon}{2}\alpha^2 \;+\; \frac{1}{2\varepsilon}\beta^2,$$

for any $\varepsilon > 0$ we can get

$$\int_U |Du||u|\, dx \;\leq\; \frac{\varepsilon}{2}\|Du\|^2_{L^2(U)} \;+\; \frac{1}{2\varepsilon}\|u\|^2_{L^2(U)},$$

and so by taking γ sufficiently large, we get

$$\begin{aligned} B[u,u] \;\geq\; \theta\, \|Du\|^2_{L^2(U)} \;-\; \||b|\|_{L^\infty(U)}\|Du\|_{L^2(U)}\|u\|_{L^2(U)} \;-\; \|c\|_{L^\infty(U)}\|u\|^2_{L^2(U)} \\ \geq\; \frac{\theta}{2}\|Du\|^2_{L^2(U)} \;-\; \gamma\|u\|^2_{L^2(U)}. \end{aligned}$$

This gives

$$B_\gamma[u,u] \;\geq\; \frac{\theta}{2}\|Du\|_{L^2(U)} \;\geq\; \theta'\|u\|_{H^1(U)}$$

Now apply Lax-Milgram.