Lecture 2: Manifolds and Riemannian metrics.

Definition. The **tangent space to Ω at** p, also known as the space of derivations at p or the space of directional derivatives at p is an n dimensional vector space denoted by $T_p(\Omega)$.

Properties: 1. If $x = (x_1, \ldots, x_n)$ are coordinates on Ω then a basis for $T_p(\Omega)$ is

$$\frac{\partial}{\partial x_1}\bigg|_p, \;\ldots,\; \frac{\partial}{\partial x_n}\bigg|_p.$$

The general tangent vector has the form

$$a_1 \frac{\partial}{\partial x_1}\bigg|_p + \ldots + a_n \frac{\partial}{\partial x_n}\bigg|_p.$$

(We can change to any other coordinates using the chain rule from last time.)
2. We can characterize derivations at p abstractly: a derivation ω at p is a linear map from $C^1(\Omega)$ to $\mathbb{R}$ which satisfies the Leibnitz property

$$\omega(fh) \;=\; \omega(f)\,h(p) \;+\; f(p)\,\omega(h).$$

3. If $\gamma : (-\varepsilon, \varepsilon) \to \Omega$ is a C^1 curve with $\gamma(0) = p$, then the directional derivative along γ at p is a derivation. It is defined by

$$\gamma'(0)\, f \;:=\; \frac{d}{dt}\bigg|_0 (f(\gamma(t))) \;=\; \sum_{i=1}^n (x_i \circ \gamma)'(0) \frac{\partial f}{\partial x_i}(p).$$

Hence with respect to the x-coordinates,

$$\gamma'(0) \;=\; \sum_{i=1}^n (x_i \circ \gamma)'(0) \frac{\partial}{\partial x_i}\bigg|_p.$$

Remark 1. A choice of coordinates x on Ω identifies $T_p(\Omega)$ with $\mathbb{R}^n$ by

$$\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\bigg|_p \;\to\; (a_1, \ldots, a_n).$$

If the coordinates x are Euclidean coordinates on $\Omega \subset \mathbb{R}^n$ then this identifies the directional derivative $\gamma'(t)$ with the vector calculus definition of the tangent vector $\gamma'(t)$ as a vector in $\mathbb{R}^n$.

Remark 2. While $dx_i|_p$ is always the same regardless of the other coordinate functions, $\partial/\partial x_i|_p$ depends on all the functions $(x_1, \ldots, x_n)$, since it is actually the directional derivative along the curve given by holding the other $x_j s$ constant.

Remark 3. In fact a tangent vector can be defined as an equivalence class of curves, that is maps from $\mathbb{R}$ to Ω, just as a cotangent vector can be defined as an equivalence class of functions from Ω to $\mathbb{R}$.

Definition. If V is a finite dimensional vector space, the **dual space** V^* is the space of linear maps $\theta : V \to \mathbb{R}$. It is a vector space of the same dimension as V. We have $V^{**} = V$.
Just as tangent vectors at p act on functions, they also act on cotangent vectors at p via

$$\left(\sum_i a_i \frac{\partial}{\partial x_i}\bigg|_p \right) df|_p \ = \ \sum_i a_i \frac{\partial f}{\partial x_i}(p).$$

This identifies the tangent space and the cotangent space as dual to each other.

Manifolds.
Definition. The topological space M is *locally Euclidean of dimension* n if each point has a neighborhood which is homeomorphic to an open set in $\mathbb{R}^n$.

Definition. A *topological manifold* M of dimension n is a topological space which satisfies the following properties.
(i) M is Hausdorff
(ii) M is locally Euclidean of dimension n
(iii) M has a countable basis of open sets.

Definition. A smooth (C^∞) structure on a topological manifold M is a family $\mathcal{U} = (U_\alpha, \phi_\alpha)$ of *coordinates charts* where for each α, U_α is an open set in M, (*coordinate neighborhood*) $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto its image (*coordinate map*) such that
(1) The sets U_α cover M.
(2) For any α, β the map $\phi_\beta \phi_\alpha^{-1} \ : \ \phi_\alpha(U_\alpha \cap U_\beta) \ \to \ \phi_\beta(U_\alpha \cap U_\beta)$ is smooth.

Definition. A *smooth manifold* is a topological manifold with a smooth structure.

Definition. If M is a smooth manifold, and $U \subset M$ is open, a map $f : U \to \mathbb{R}$ is smooth (C^∞) if for every coordinate chart (U_α, ϕ_α) in the smooth structure on M, the map

$$f \circ \phi_\alpha : \phi_\alpha(U\alpha \cap U) \to \mathbb{R}$$

is smooth.

We will just draw a picture of what it means for a map from one smooth manifold to another to be smooth. It is an exercise to define the general smooth coordinate chart on a smooth manifold M.

Definition. If M is a smooth manifold of dimension n, the tangent space to M at p is the space of derivations at p as defined above. It is an n-dimensional manifold. If $x = (x_1, \ldots, x_n)$ are smooth coordinates on M on a neighborhood of p, then it is spanned by

$$\frac{\partial}{\partial x_1}\bigg|_p, \ldots, \frac{\partial}{\partial x_n}\bigg|_p.$$

Definition. If M is a smooth manifold, a **Riemannian metric** on M is a symmetric positive definite bilinear form $g(p) = \langle , \rangle|_p$ defined on $T_p(M)$ for each $p \in M$, which varies smoothly in the sense that for any coordinates chart (U, x) on M, the matrix elements

$$g_{ij}(p) \;:=\; \left\langle \frac{\partial}{\partial x_i}\bigg|_p , \frac{\partial}{\partial x_j}\bigg|_p \right\rangle_p$$

are smooth on U.

The important thing about the metric is that it is exactly what we need to measure the length of curves and the angle between curves. In particular, if $\gamma : [a, b] \to M$ is a C^1 curve, then the length of γ is

$$L(\gamma) \;=\; \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}}\, dt.$$

Example 1. If $M = \mathbb{R}^n$ and x are Euclidean coordinates, then the standard metric is

$$\left\langle \frac{\partial}{\partial x_i}\bigg|_p , \frac{\partial}{\partial x_j}\bigg|_p \right\rangle_p \;=\; \delta_{ij}.$$

When tangent vectors are identified with elements of $\mathbb{R}^n$ via their coordinates, this gives the usual inner product on $\mathbb{R}^n$,

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle \;=\; \sum_i a_i b_i.$$

Example 2. An operator $L : C^\infty(M) \to C^\infty(M)$ is a **Second order elliptic linear partial differential operator** if for every coordinate chart (Ω, x) in the smooth structure and all functions u supported on Ω it has the form

$$Lu(p) \;=\; \sum_{i,j} a^{ij}(p) u_{x_i x_j}(p) \;+\; \sum_i b^i(p) u_{x_i}(p) \;+\; c(p) u(p).$$

for some smooth coefficients a^{ij}, b^i, c with the usual (pointwise) ellipticity condition on a^{ij}. Then writing $a_{ij}(p)$ for the inverse of the matrix $a^{ij}(p)$, we have that

$$\left\langle \frac{\partial}{\partial x_i}\bigg|_p , \frac{\partial}{\partial x_j}\bigg|_p \right\rangle_p \;:=\; a_{ij}(p)$$

gives a well defined metric on M. This metric comes into the study of the operator L. In particular for the evolution equation

$$\frac{\partial^2}{\partial t^2} u(p, t) \;=\; Lu(p, t),$$

singularities in the initial data $u(p,0)$ propagate along geodesics (length minimizing curves).

Example 3. Suppose that M is a set in $\mathbb{R}^3$ of the form $f(x_1, x_2, x_3) = 0$, where f is a smooth function. Suppose also that $df(p) \neq 0$ on M. Then M is a smooth surface, that is smooth manifold of dimension 2. An example is the 2-sphere $x^2+y^2+z^2-1 = 0$. (In fact its coordinate maps can be chosen to be graphs over open subsets of the coordinate planes.)

Because this surface is embedded in $\mathbb{R}^3$, it inherits a natural metric from the usual metric $\langle\ ,\ \rangle$ on $\mathbb{R}^3$. Indeed, if $\gamma, \mu : [-\varepsilon, \varepsilon] \to M$ are curves with $\gamma(0) = \mu(0) = p$, then $\gamma'(0)$ and $\mu'(0)$ define tangent vectors to M at p, and to $\mathbb{R}^3$ at p. We can define

$$\langle \gamma'(0), \mu'(0) \rangle|_{M,p} \;=\; \langle \gamma'(0), \mu'(0) \rangle|_{\mathbb{R}^3,p}.$$

Remark. If you take a piece of paper and bend it in space without tearing or stretching it, the metric it inherits from $\mathbb{R}^3$ is always the same. The metric describes the intrinsic geometry of the surface, but not how it is embedded in the space.

Next lecture: Surface theory: The area element dA. The Laplacian $\Delta = \nabla \cdot \nabla$. The curvature K. Conformal change. Gauss Bonnet.