

Lecture 20. The Fredholm Alternative.

Lax-Milgram Theorem. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Assume that*

$$B : H \times H \rightarrow \mathbb{R}$$

is a bilinear mapping such that there exist constants $\alpha, \beta > 0$ with

$$(1) \quad |B[u, v]| \leq \alpha \|u\| \|v\|, \quad u, v \in H$$

and

$$(2) \quad \beta \|u\|^2 \leq B[u, u] \quad u \in H.$$

Then if f is a continuous linear functional on H , there exists a unique element $u \in H$ such that

$$B[u, v] = f(v), \quad \text{for all } v \in H.$$

Proof. By (1), the linear map

$$v \rightarrow B[u, v]$$

is bounded and hence by the Riesz Representation Theorem (RRT) there exists a unique element Au such that

$$B[u, v] = \langle Au, v \rangle, \quad \text{for all } v \in H.$$

However the map $u \rightarrow Au$ is easily checked to be linear using the bilinearity of B . We claim that A is a linear isomorphism. Indeed

$$(3) \quad \beta \|u\|^2 \leq B[u, u] = \langle Au, u \rangle \leq \|Au\| \|u\|$$

so if $Au = 0$ then the right hand side vanishes and so $u = 0$. Hence A is one-to-one. Furthermore dividing by $\|u\|$ we have

$$\beta \|u\| \leq \|Au\|,$$

and so the image AH is closed and A^{-1} is bounded on AH . However we see that if $w \in AH^\perp$ then $\langle Aw, w \rangle = 0$ and so from (3) we have $w = 0$. Hence A^{-1} is a bounded linear map on H .

Now by the RRT there exists w with

$$f(v) = \langle w, v \rangle \quad \text{for all } v \in H.$$

But then

$$f(v) = \langle w, v \rangle = B[A^{-1}w, v].$$

Fredholm Alternative. Let H be a Hilbert space and let $K : H \rightarrow H$ be a compact linear operator. Then

- (i). The null space $N(I - K)$ is finite dimensional.
- (ii). The range $R(I - K)$ is closed.
- (iii). $R(I - K) = N(I - K^*)^\perp$.
- (iv). $N(I - K) = \{0\} \Leftrightarrow R(I - K) = H$.
- (v). $\dim N(I - K) = \dim N(I - K^*)$.

Remark. K^* is the adjoint defined by

$$\langle Ku, v \rangle = \langle u, K^*v \rangle, \quad \text{for all } u, v \in H.$$

Let H be a Hilbert space with inner product (\cdot, \cdot) . Let K be a compact operator on H . (If u_j is a bounded sequence in H then there exists a subsequence u_{j_k} such that Ku_{j_k} converges in H .)

Lemma. If u_j converges to u weakly in H then $Ku \rightarrow Ku$ in H .

Proof. By the *uniform boundedness principle*, $\|u_j\|$ is uniformly bounded. Now we see that any convergent subsequence of Ku_j must converge to Ku . Indeed, suppose that Ku_{j_k} converges to w . Then for every v ,

$$(Ku_{j_k}, v) \rightarrow (w, v),$$

but

$$(Ku_{j_k}, v) = (u_{j_k}, K^*v) \rightarrow (u, K^*v) = (Ku, v).$$

Hence $w = Ku$. Now since K is compact, every subsequence of Ku_j has a convergent subsequence, which by the previous argument converges to Ku . This shows that Ku_j converges to Ku . (Else we can construct a subsequence which does not have a subsequence converging to Ku .)

Lemma. If K is compact, so is K^* .

Proof. If u_j is a bounded sequence in H , choose a weakly convergent subsequence, u_{j_k} converging weakly to u , and hence bounded.

$$(*) \ \|K^*u_{j_k} - K^*u\|^2 = (K^*(u_{j_k} - u), K^*(u_{j_k} - u)) = (KK^*(u_{j_k} - u), (u_{j_k} - u)).$$

However, KK^* is compact, so $KK^*(u_{j_k} - u)$ converges to zero in H by the previous lemma. Hence $(*)$ tends to zero.

Exercise. Show that if H is a Hilbert space, A is a bounded operator on H and B is a compact operator, then AB and BA are compact.

Proof of the Fredholm Alternative. (i). If $\dim N(I - K) = \infty$ then there exists an infinite orthonormal set u_j with $Ku_j = u_j$. Then Ku_j has no convergent subsequence because $\|Ku_j - Ku_k\| = \|u_j - u_k\| = \sqrt{2}$ for all j, k .

(ii). We claim that for some $\gamma > 0$,

$$(**) \quad \|(I - K)u\| \geq \gamma\|u\|, \quad \text{all } u \in N(I - K)^\perp.$$

Otherwise, there exists $u_j \in N(I - K)^\perp$ with $\|u_j\| = 1$ but $(I - K)u_j \rightarrow 0$. However, Ku_j has a convergent subsequence $Ku_{j_k} \rightarrow w$, and hence $u_{j_k} \rightarrow w$ and $w \in N(I - K)^\perp$. However, $(I - K)w = \lim(I - K)u_{j_k} = 0$, so $w \in N(I - K)$. Hence $w = 0$. But since $\|u_j\| = 1$ this is a contradiction. Now we see that $(I - K)$ restricted to $N(I - K)^\perp$ is a one-to-one map onto $R(I - K)$ and its inverse is continuous. Hence it is a linear isomorphism and $R(I - K)$ is complete and hence closed in H .

(iii). In general, for $A : H \rightarrow H$ bounded, $\overline{R(A)} = N(A^*)^\perp$. Indeed, suppose $u \in R(A)$ and $v \in N(A^*)$. Then $u = Aw$ for some w and

$$(u, v) = (Aw, v) = (w, A^*v) = 0.$$

Hence $R(A) \subset N(A^*)^\perp$. Now suppose $u \in R(A)^\perp$. Then for every w ,

$$0 = (Aw, u) = (w, A^*u).$$

Hence $u \in N(A^*)$. Thus $R(A)^\perp \subset N(A^*)$, and so $N(A^*)^\perp \subset \overline{R(A)}$ and we have equality. In our case, Since $R(I - K)$ is closed, we get $R(I - K) = N(I - K^*)^\perp$.