MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 23. Regularity.

$$Lu = \sum_{i,j}(a^{ij}u_{x_i})_{x_j} + \sum_i b^i u_{x_i} + cu.$$

$$B[u,v] = \int_U \left(\sum_{i,j}(a^{ij}u_{x_i})_{x_j} + \sum_i b^i u_{x_i} + cu\right) dx$$

We say $u \in H_0^1(U)$ is a weak solution of $Lu = f$ if

$$B[u,v] = \int fv \qquad \text{for all } v \in H_0^1(U).$$

We say $v \in H_0^1(U)$ is a weak solution of $L^*v = f$ if

$$B^*[v,u] = B[u,v] = \int fu \qquad \text{for all } u \in H_0^1(U).$$

Theorem. *(Second Existence Theorem.) Suppose U is a bounded open set in $\mathbb{R}^n$ with C^1 boundary. Precisely one of the following two statements holds (this is called the Fredholm alternative):*

Statement 1: *For each $f \in L^2(U)$ there exists a unique solution $u \in H_0^1(U)$ of the boundary value problem*

(1)
$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Statement 2: *There exists $u \in H_0^1(U)$ not equal to the zero function solving the homogeneous problem*

(2)
$$\begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Furthermore, the dimension of the subpace $N \subset H_0^1(U)$ of weak solutions of (2) is finite and equal to the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions of

(3)
$$\begin{cases} L^*u = 0 & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases}$$

and the boundary-value problem (*) has a weak solution if and only if

(4)
$$\int_U fv\,dx = 0, \qquad \text{for all } v \in N^*.$$

Theorem. *(Third Existence Theorem.) (i) There exists a countable (or finite) set Σ (called the real spectrum of L) such that the boundary value problem*

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.

(ii) If Σ is infinite then $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$ with $\lambda_k \to \infty$.

Proof. We can choose $\gamma > 0$ such that

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique weak solution for all $\mu \geq \gamma$ and $f \in L^2(U)$. Then when $\mu = \gamma$ we denoted the solution f by Ku. We showed that K is a compact operator and for $u \in H_0^1(U)$ and $f \in L^2(U)$,

$$Lu + \gamma u = f \;\Leftrightarrow\; u = Kf.$$

But then for $u \in H_0^1(U)$ and $f \in L^2(U)$,

$$Lu = \lambda u + f \;\Leftrightarrow\; Lu + \gamma u = (\gamma+\lambda)u + f \;\Leftrightarrow\; u = (\gamma+\lambda)Ku + Kf \;\Leftrightarrow\; u - (\gamma+\lambda)Ku = Kf.$$

We see that if $I - (\gamma+\lambda)K$ is invertible, then we always have a unique weak solution. However

$$I - (\gamma+\lambda)K \text{ invertible } \Leftrightarrow\; \frac{1}{\gamma+\lambda} \notin \sigma(K).$$

Since $\sigma(K)$ is either finite or a sequence tending to zero, we see that $1/(\gamma+\lambda)$ is either finite or equals a sequence tending to infinity. When $1/(\gamma+\lambda) \in \sigma(K)$, then the operator

$$K \;-\; \frac{I}{\gamma+\lambda}$$

has non-trivial null space, and so there are non-trivial weak solutions $u \in H_0^1(U)$ to the equation $Lu = \lambda u$. In this case λ is called an **eigenvalue** of L.

Regularity.

Theorem. *Assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, $f \in L^2(U)$ and $u \in H^1(U)$, and that u is a weak solution of the elliptic PDE*

$$Lu = f \qquad\qquad \text{in } U.$$

Then

$$u \in H^2_{\mathrm{loc}}(U)$$

and for each $V \subset\subset U$ *there exists* $C = C(V, U, L)$ *such that*

$$\|u\|_{H^2(V)} \;\leq\; C(\|f\|_{L^2(U)} \;+\; \|u\|_{L^2(U)}).$$

Proof. We start by showing that for $W \subset\subset U$, if u is a weak solution of $Lu = f$ on U, then

$$\int_W |Du|^2\,dx \;\leq\; C\left(\int_U |f|^2\,dx \;+\; \int_U |u|^2\,dx\right).$$

To see this, choose $\zeta \in C_c^\infty(U)$ with $\zeta = 1$ on W. Note that since $Lu = f$ weakly, we can set $v = \zeta^2 u$ in the equation

$$B[u, v] \;=\; \int_U fv\,dx,$$

to get

$$\begin{aligned}\int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j}\,dx \;&=\; \int_U \left(f - \sum_i b^i u_{x_i} - cu\right) v\,dx \\ &\leq\; C\left(\int_U f^2\,dx \;+\; \int_U \zeta|uDu|\,dx \;+\; \int_U |u|^2\,dx\right),\end{aligned}$$

but

$$\begin{aligned}\int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j}\,dx \;&=\; \int_U \zeta^2 \sum_{i,j} a^{ij} u_{x_i} u_{x_j}\,dx \;+\; \int_U \sum_{i,j} a^{ij} u_{x_i} u \zeta \zeta_{x_j}\,dx \\ &\geq\; \theta \int_U \zeta^2 |Du|^2\,dx \;-\; C\int_U \zeta|uDu|\,dx.\end{aligned}$$

Hence

$$\theta \int_U \zeta^2 |Du|^2\,dx \;\leq\; C'\left(\int_U f^2\,dx \;+\; \int_U \zeta|uDu|\,dx \;+\; \int_U |u|^2\,dx\right),$$

But by Cauchy's theorem,

$$\int_U \zeta|uDu|\,dx \;\leq\; \varepsilon\int_U \zeta^2|Du|^2\,dx \;+\; \frac{1}{4\varepsilon}\int_U |u|^2\,dx,$$

so by choosing ε sufficiently small, we get

$$\int_U \zeta^2|Du|^2\,dx \;\leq\; C\left(\int_U |f|^2\,dx \;+\; \int_U |u|^2\,dx\right).$$