Lecture 24. Regularity.

$$Lu \;=\; \sum_{i,j}(a^{ij}u_{x_i})_{x_j} \;+\; \sum_i b^i u_{x_i} \;+\; cu.$$

$$B[u,v] \;=\; \int_U \left(\sum_{i,j}(a^{ij}u_{x_i})_{x_j} \;+\; \sum_i b^i u_{x_i} \;+\; cu \right) dx$$

We say $u \in H_0^1(U)$ is a weak solution of $Lu = f$ if

$$B[u,v] = \int fv \qquad \text{for all } v \in H_0^1(U).$$

Theorem. *Assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, $f \in L^2(U)$ and $u \in H^1(U)$, and that u is a weak solution of the elliptic PDE*

$$Lu = f \qquad\qquad \textit{in } U.$$

Then

$$u \in H^2_{\text{loc}}(U)$$

Last time: for $W \subset\subset U$, if u is a weak solution of $Lu = f$ on U, then

$$\int_W |Du|^2\,dx \;\leq\; C\left(\int_U |f|^2\,dx \;+\; \int_U |u|^2\,dx\right).$$

Now we see that we just need to consider the case $b^i = c = 0$. Indeed, $Lu = f$ weakly in U can be written as

$$-\sum_{i,j}(a^{ij}u_{x_i})_{x_j} \;=\; f \;-\; \sum_i b^i u_{x_i} \;-\; cu \;=\; \tilde{f}$$

weakly in U, and $\tilde{f} \in L^2(U)$. Moreover, choosing W with $V \subset\subset W \subset\subset V$, we have

$$\|\tilde{f}\|_{L^2(W)} \;\leq\; C(\|f\|_{L^2(U)} \;+\; \|u\|_{L^2(U)}).$$

By working on the smaller set W, it thus suffices to prove the result when $b^i = c = 0$.

The tool we will need to prove this result is the *difference quotient*. For $u : U \to \mathbb{R}$, define

$$D_i^h u(x) \;=\; \frac{u(x+he_i) - u(x)}{h}, \qquad\qquad x \in U,\; 0 < |h| < \text{dist}(x, \partial U).$$

Lemma. *(a). If* $u, v \in L^2(U)$*,* v *is supported in* V*, and* $0 < |h| < \operatorname{dist}(V, \partial U)$*, then*

$$\int_U u D_i^{-h} v \, dx \;=\; -\int_U (D_i^h u) v \, dx.$$

(b). In general, $D_i^h(uv) \;=\; (D_i^h u)v \;+\; (T_i^h u)(D_i^h v)$*, where*

$$T_i^h u(x) \;=\; u(x + he_i).$$

Theorem. *(i) Suppose* $1 \le p < \infty$ *and* $u \in W^{1,p}(U)$*. Then for each* $V \subset U$ *there exists a constant* C *such that*

$$\|D^h u\|_{L^p(V)} \;\le\; C\|Du\|_{L^p(U)},$$

if $0 < |h| < \frac{1}{2}\,dist(V, \partial U)$
(ii) Conversely, suppose $1 < p < \infty$ *and* $u \in L^p(U)$ *and there exists a constant* C *such that*

(1) $$\|D^h u\|_{L^p(V)} \;\le\; C,$$

for $0 < |h| < \varepsilon < \frac{1}{2}\,dist(V, \partial U)$*. Then*

$$u \in W^{1,p}(V) \qquad \text{and} \qquad \|Du\|_{L^p(V)} \;\le\; C.$$

Proof of (ii). Equation (1) implies that for each i,

$$\sum_{0<|h|<\varepsilon} \|D_i^{-h} u\|_{L^p(V)} \;<\; \infty.$$

and hence since $1 < p < \infty$ and $L^p(V)$ is reflexive, there exists a sequence $h_k \to 0$ and $v_i \in L^p(V)$ with

$$D_u^{-h_k} u \;\to\; v_i \qquad \text{weakly in } L^p(V).$$

But then if $\phi \in C_c^\infty(V)$, we have

$$\int_V u\phi_{x_i}\,dx \;=\; \int_U u\phi_{x_i}\,dx \;=\; \lim_{k\to\infty}\int_U u D_i^{h_k}\phi\,dx \;=\; -\lim_{k\to\infty}\int_U (D_i^{-h_k}u)\phi\,dx \;=\; -\int_V v_i\phi\,dx.$$

Hence $v_i = u_{x_i}$ in the weak sense and so $Du \in L^p(V)$. Since $u \in L^p(V)$ we deduce $u \in W^{1,p}(V)$.

Back to the proof of the Theorem. We have $a^{ij} \in C^1(U)$ and $b^i, c \in L^\infty(U)$, and $u \in H_0^1(U)$ and $f \in L^2(U)$ and u is a weak solution of

$$-\sum_{i,j}(a^{ij}u_{x_i})_{x_j} \;=\; f, \qquad x \in U,$$

and we wish to show that $u \in H^2_{\rm loc}(U)$ and for each $V \subset\subset U$

$$\|u\|_{H^2(V)} \;\le\; C(\|f\|_{L^2(U)} \;+\; \|u\|_{L^2(U)}).$$

We choose $\zeta \in C_c^\infty(U)$ with $\zeta = 1$ on V, and support in W. Set

$$v \;=\; -D_k^{-h}(\zeta^2 D_k^h u).$$

Then

$$\int_U a^{ij} u_{x_i} v_{x_j}\, dx \;=\; \int_U f v\, dx.$$

Hence for every $\varepsilon > 0$ we get

$$(*) \qquad \int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j}\, dx \;=\; \int_U f v\, dx \;\le\; \varepsilon \int_U |v|^2\, dx \;+\; \frac{C}{\varepsilon}\int_U |f|^2\, dx,$$

but

$$\int_U |v|^2\, dx \;\le\; C\int_U |D(\zeta^2 D_k^h u)|^2\, dx \;\le\; C\int_U \zeta^2 |D_k^h Du|^2\, dx \;+\; C\int_U \zeta^2 |D_k^h u|^2\, dx$$
$$\le\; C\int_U \zeta^2 |D_k^h Du|^2\, dx \;+\; C\int_W |Du|^2\, dx$$

So

$$(*) \quad \int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j}\, dx \;\le\; C\varepsilon\int_U \zeta^2 |D_k^h Du|^2\, dx + \frac{C}{\varepsilon}\int_W |Du|^2\, dx + C\int_U |f|^2\, dx,$$

However, using $D_k^h(au) = (T_k^h a) D_k^h u + (D_k^h a)u$, where $T_k^h a(x) = a(x + he_k)$. The left hand side of (*) is

$$\int_U \sum_{i,j} (D_k^h(a^{ij} u_{x_i}))(\zeta^2 D_k^h u)_{x_j}\, dx$$
$$= \int_U \sum_{i,j,k} \zeta^2 (T_k^h a^{ij})(D_k^h u_{x_i})(D_k^h u_{x_j})\, dx \;+\; \sum_{i,j} (T_h a^{ij})(D_k^h u_{x_i})(D_k^h u)\zeta\zeta_{x_j}\, dx$$
$$+ \sum_{i,j} (D_k^h a^{ij}) u_{x_i} \zeta^2 D_k^h u_{x_j}\, dx \;+\; \sum_{i,j} (D_k^h a^{ij}) u_{x_i} (D_k^h u) 2\zeta\zeta_{x_j}\, dx$$
$$\ge \theta\int_U \zeta^2 |D_k^h Du|^2\, dx$$
$$-\; C\left(\int_U \zeta |D_k^h Du||D_k^h u|\, dx \;+\; \int_U \zeta |D_k^h Du||Du|\, dx \;+\; \int_U \zeta |D_k^h u||Du|\, dx\right)$$
$$\ge \theta\int_U \zeta^2 |D_k^h Du|^2\, dx \;-\; \left(\varepsilon\int_U \zeta^2 |D_k^h Du|^2\, dx \;+\; \frac{C}{\varepsilon}\int_W |Du|^2\, dx\right).$$

Hence we get

$$\theta\int_U \zeta^2 |D_k^h Du|^2\, dx \;\le\; C\varepsilon\int_U \zeta^2 |D_k^h Du|^2\, dx + \frac{C}{\varepsilon}\int_W |Du|^2\, dx \;+\; C\int_U |f|^2\, dx.$$

Choosing $C\varepsilon < \theta$ we get

$$\|\zeta D_k^h Du\|^2_{L^2(U)} \;\le\; C\left(\int_W |Du|^2\, dx \;+\; \int_U |f|^2\, dx\right) \;\le\; C'\left(\|f\|^2_{L^2(U)} \;+\; \|u\|_{L^2(U)}\right).$$