Lecture 25. Regularity.

$$Lu \;=\; -\sum_{i,j}(a^{ij}u_{x_i})_{x_j}$$

$u \in H_0^1(U)$ is a weak solution of $Lu = f$ if

$$B[u,v] \;=\; \int_U \sum_{i,j} a^{ij}u_{x_i}v_{x_j}\,dx = \int fv \qquad \text{for all } v \in H_0^1(U).$$

Theorem. *Assume $a^{ij} \in C^1(U)$, $f \in L^2(U)$ and $u \in H^1(U)$, and that u is a weak solution of the elliptic PDE*

$$Lu = f \qquad\qquad \text{in } U.$$

Then

$$u \in H^2_{\mathrm{loc}}(U),$$

and

$$\|u\|_{H^2(U)} \;\le\; C\left(\|f\|_{L^2(U)} \;+\; \|u\|_{L^2(U)}\right).$$

- we showed for $W \subset\subset U$,

$$\int_W |Du|^2\,dx \;\le\; C\left(\int_U |f|^2\,dx \;+\; \int_U |u|^2\,dx\right).$$

- *Difference quotient.*

$$D_i^h u(x) \;=\; \frac{u(x+he_i)-u(x)}{h}, \qquad\qquad x \in U,\; 0<|h|<\mathrm{dist}(x,\partial U).$$

- $D_i^h(uv) \;=\; (D_i^h u)v \;+\; (T_i^h u)(D_i^h v)$, where $T_i^h u(x) \;=\; u(x+he_i)$.
- If $V \subset U$, and $0<|h|<\mathrm{dist}(V,\partial U)$,

$$\int_U uD_i^{-h}v\,dx \;=\; -\int_U (D_i^h u)v\,dx.$$

- $1 \le p < \infty$, $u \in W^{1,p}(U)$, $V \subset U$, $0<|h|<\frac{1}{2}\mathrm{dist}(V,\partial U)$.

$$\|D^h u\|_{L^p(V)} \;\le\; C\|Du\|_{L^p(U)}, \qquad\qquad C = C(p).$$

• $1 < p < \infty$, $u \in L^p(U)$. Suppose there exists C such that

$$\|D^h u\|_{L^p(V)} \;\le\; C,$$

when $0 < |h| < \varepsilon < \frac{1}{2}\mathrm{dist}(V, \partial U)$. Then

$$u \in W^{1,p}(V) \qquad \text{and} \qquad \|Du\|_{L^p(V)} \;\le\; C.$$

We choose $\zeta \in C_c^\infty(U)$ with $\zeta = 1$ on V, and support in W. Set

$$v \;=\; -D_k^{-h}(\zeta^2 D_k^h u).$$

Then

$$\int_U a^{ij} u_{x_i} v_{x_j}\, dx \;=\; \int_U f v\, dx.$$

Hence for every $\varepsilon > 0$ we get

$$(*) \qquad \int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j}\, dx \;=\; \int_U f v\, dx \;\le\; \varepsilon \int_U |v|^2\, dx \;+\; \frac{C}{\varepsilon}\int_U |f|^2\, dx,$$

but

$$\begin{aligned}\int_U |v|^2\, dx \;\le\; C\int_U |D(\zeta^2 D_k^h u)|^2\, dx \;&\le\; C\int_U \zeta^2 |D_k^h Du|^2\, dx \;+\; C\int_U \zeta^2 |D_k^h u|^2\, dx \\ &\le\; C\int_U \zeta^2|D_k^h Du|^2\, dx \;+\; C\int_W |Du|^2\, dx\end{aligned}$$

So

$$(*) \;\int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j}\, dx \;\le\; C\varepsilon \int_U \zeta^2 |D_k^h Du|^2\, dx + \frac{C}{\varepsilon}\int_W |Du|^2\, dx + C\int_U |f|^2\, dx,$$

However, using $D_k^h(au) = (T_k^h a) D_k^h u + (D_k^h a) u$, where $T_k^h a(x) = a(x + he_k)$. The left hand side of (*) is

$$\begin{aligned}&\int_U \sum_{i,j} (D_k^h(a^{ij} u_{x_i}))(\zeta^2 D_k^h u)_{x_j}\, dx \\ &\quad= \int_U \sum_{i,j,k} \zeta^2 (T_k^h a^{ij})(D_k^h u_{x_i})(D_k^h u_{x_j})\, dx \;+\; \sum_{i,j} (T_h a^{ij})(D_k^h u_{x_i})(D_k^h u)\zeta\zeta_{x_j}\, dx \\ &\qquad\quad + \sum_{i,j} (D_k^h a^{ij}) u_{x_i} \zeta^2 D_k^h u_{x_j}\, dx \;+\; \sum_{i,j} (D_k^h a^{ij}) u_{x_i} (D_k^h u) 2\zeta\zeta_{x_j}\, dx \\ &\quad \ge \theta \int_U \zeta^2 |D_k^h Du|^2\, dx \\ &\qquad - C\left(\int_U \zeta |D_k^h Du||D_k^h u|\, dx \;+\; \int_U \zeta |D_k^h Du||Du|\, dx \;+\; \int_U \zeta|D_k^h u||Du|\, dx\right) \\ &\quad \ge \theta\int_U \zeta^2 |D_k^h Du|^2\, dx \;-\; \left(\varepsilon\int_U \zeta^2 |D_k^h Du|^2\, dx \;+\; \frac{C}{\varepsilon}\int_W |Du|^2\, dx\right).\end{aligned}$$

Hence we get

$$\theta \int_U \zeta^2 |D_k^h Du|^2 \, dx \;\le\; C\varepsilon \int_U \zeta^2 |D_k^h Du|^2 \, dx \;+\; \frac{C}{\varepsilon} \int_W |Du|^2 \, dx \;+\; C \int_U |f|^2 \, dx.$$

Choosing $C\varepsilon < \theta$ we get

$$\|\zeta D_k^h Du\|_{L^2(U)}^2 \;\le\; C\left(\int_W |Du|^2 \, dx \;+\; \int_U |f|^2 \, dx\right) \;\le\; C'\left(\|f\|_{L^2(U)}^2 \;+\; \|u\|_{L^2(U)}\right).$$