MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 4: Sobolev Spaces.

Last time We defined the volume form. Note that for surfaces,

$$dA_{e^{2u}g} \;=\; e^{2u} dA_g.$$

We defined the curvature $K(p)$ of a surface with a metric at a point p.

Remark. For a closed surface there is a famous formula called the Gauss-Bonnet formula which states that

$$\int_M K\, dA \;=\; 4\pi\chi(M) \;=\; 4\pi(2-2g). \tag{3}$$

Here g is the genus of the surface, i.e. the number of handles. This is analogous to the result that the sum of the exterior angles of a polygon add up to 2π. We will not use this formula, but it is useful to know.

Remark. If $\alpha > 0$ is constant then

$$K_{\alpha^2 g} \;=\; \alpha^{-2} K_g.$$

Support. If $f : M \to \mathbb{R}$, the **support** of f is

$$\operatorname{support}(f) \;=\; \overline{\{x : f(x) \neq 0\}}.$$

Partition of unity. Given open sets U_α which cover a smooth manifold M, we can find a **partition of unity subordinate to** $\{U_\alpha\}$**.** This means a set of smooth functions $\chi_\alpha : M \to [0,1]$ such that χ_α is supported in U_α and

$$\sum_\alpha \chi_\alpha \;=\; 1 \text{ on } M.$$

The Laplacian. If f is a smooth function on the manifold M with metric g, then we can define the **gradient of** f, ∇f, which is a smooth vector field on M. (A vector field is a choice of vector in T_pM for each $p \in M$.) By definition

$$\nabla f(p) \;=\; \sum_{i,j} g^{ij}(p) \frac{\partial f}{\partial x_i}\bigg|_p \frac{\partial}{\partial x_j}\bigg|_p .$$

Here, g^{ij} is the inverse of the metric g in x coordinates. It is the dual vector to $df(p) \in T^*M$. If F is a vector field on M, we define the **divergence of** F, $\nabla \cdot F$ by

$$\nabla \cdot \sum_i F_i \frac{\partial}{\partial x_i} \;=\; \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\; F_i\right).$$

The gradient and divergence are independent of the choice of coordinates. We define

$$\Delta f \;=\; \nabla\cdot\nabla f \;=\; \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x_j} \;=\; \sum_{i,j} g^{ij} f_{x_i x_j} + \text{ lower order term.}$$

Theorem. . *If F is a smooth vector field on M then*
(a). $\nabla \cdot (uF) \;=\; \nabla u \cdot F \;+\; u \nabla \cdot F.$
(b). **Divergence Theorem**

$$\int_M \nabla \cdot F \, dV \;=\; 0.$$

(c).

$$\int_M u \nabla \cdot F \, dV \;=\; -\int \nabla u \cdot F \, dV.$$

(d).

$$\int_M u \Delta v \, dV \;=\; -\int_M \nabla u \cdot \nabla v \, dV.$$

Proof of (b). Choose a partition of unity χ_α subordinate to coordinate neighborhoods (U_α, ϕ_α). This enables us to work in coordinates. We have

$$\int_M \nabla \cdot F \, dV \;=\; \sum_\alpha \int_M \nabla \cdot \xi_\alpha F \, dV.$$

We show that each term in this sum vanishes. Indeed, writing $\phi_\alpha = (x_1, \dots, x_n)$, and

$$F \;=\; \sum_i F_i \frac{\partial}{\partial x_i},$$

and extending $(\chi_\alpha \cdot F_i) \circ \phi_\alpha^{-1}$ to equal zero outside the set $\phi_\alpha(U_\alpha)$, we have

$$\int_M \nabla \cdot (\chi_\alpha F) \, dV \;=\; \int_{U_\alpha} \frac{\partial(\chi_\alpha \cdot F_i)}{\partial x_i} \frac{dV}{\sqrt{\det g_{ij}}} \;=\; \int_{\mathbb{R}^n} \frac{\partial(\chi_\alpha \cdot F_i)(\chi_\alpha \cdot F_i) \circ \phi_\alpha^{-1}}{\partial x_i} \, dx_1 \dots dx_n \;=\; 0.$$

Lemma. *In normal coordinates x at p,*

$$\Delta f(p) \;=\; \sum_i f_{x_i x_i}(p).$$

This only works at single point p, but nevertheless it can be useful.

Curvature formula.

$$K_{e^{2u}g} \;=\; e^{-2u} \left(-\Delta u \;+\; K_g \right).$$

Problem. We try to solve

$$e^{-2u} \left(-\Delta u \;+\; K_g \right) \;=\; \text{constant}.$$

Multiindex notation.

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

with $\alpha_j \in \mathbb{N}$ is called a **multiindex**.
If f is a smooth function on $\mathbb{R}^n$, then

$$D^\alpha f \;:=\; \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f.$$

Sobolev spaces. If $k \geq 0$ is an integer, the Sobolev norm $\| \; \|_{W^{k,p}}$ is defined on smooth functions by

$$\|f\|_{H^k} \;=\; \left(\sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |D^\alpha f|^p \, dx \right)^{1/p} .$$

The Sobolev space $W^{k,p}(\mathbb{R}^n)$ is the completion of $C^\infty(\mathbb{R}^n)$ in the $W^{k,p}$-norm. It is a Banach space.

If M is a smooth closed manifold (compact, no boundary) then we cover M by a finite collection of coordinate charts (U_α, ϕ_α) and we take a partition of unity χ_α subordinate to $\{U_\alpha\}$. Then we define a Sobolev norm $\| \; \|_{W^{k,p}}$ by

$$\|f\|_{W^{k,p}(M)} \;=\; \sum_\alpha \|f_\alpha\|_{W^{k,p}(\mathbb{R}^n)}, \qquad f_\alpha(x) \;=\; \begin{cases} 0 & x \notin \phi_\alpha(U_\alpha), \\ (\chi_\alpha \cdot f_\alpha) \circ \phi_\alpha^{-1}(x), & x \in U_\alpha. \end{cases}$$

The Sobolev space $W^{k,p}(M)$ is the completion of $C^\infty(M)$ in the norm $\| \; \|_{W^{k,p}(M)}$.
We set $H^k(M) := W^{k,2}(M)$.

Lemma. *The space $W^{k,p}(M)$ does not depend on the choice of smooth charts $(\Omega_\alpha, \phi_\alpha)$ which cover M.*