Lecture 5: Elliptic regularity.

Remark. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
Last time: If M is a smooth closed manifold (compact, no boundary) then we cover M by a finite collection of coordinate charts (U_j, ϕ_j) with $1 \leq j \leq N$. Choose open sets V_j so that $\bar{V}_j \subset U_j$, and $\bigcup_j V_j = M$. Then we define a Sobolev norm $\| \ \|_{W^{k,p}}$ by

$$\|f\|_{W^{k,p}(M)} = \left(\sum_{j=1}^{N} \sum_{|\alpha| \leq k} \int_{\phi(V_j)} |D^\alpha (f \circ \phi_j^{-1})|^p dx \right)^{1/p}.$$

The Sobolev space $W^{k,p}(M)$ is the completion of $C^\infty(M)$ in the norm $\| \ \|_{W^{k,p}(M)}$.

Lemma. *If we choose two different sets of coordinate charts (U_k, ϕ_j) and/or different sets V_j, resulting in two norms $\| \ \|_{W^{k,p}(M)}$ and $\| \ \|'_{W^{k,p}(M)}$, then there exists $c > 0$ such that for all $f \in C^\infty(M)$,*

$$\frac{1}{c}\|f\|_{W^{k,p}(M)} \leq \|f\|'_{W^{k,p}(M)} \leq c\|f\|_{W^{k,p}(M)}.$$

Hence the space $W^{k,p}(M)$ does not depend on the choice of smooth charts (U_j, ϕ_j) or V_j with $V_j \subset U_j$ which cover M.

Definition. $W^{0,p}(M) = L^p(M)$. Also $H^k(M) := W^{k,2}(M)$ can be given the structure of a **Hilbert space**. The inner product is defined on smooth functions by

$$\langle f, h \rangle = \left(\sum_{j=1}^{N} \sum_{|\alpha| \leq k} \int_{\phi_j(V_j)} (D^\alpha (f \circ \phi_j^{-1}))(D^\alpha (h \circ \phi_j^{-1})) dx \right)^{1/2}$$

Remark. We can define

$$\|f\|_{C^k(M)} = \max_{1 \leq j \leq N} \max_{|\alpha| \leq k} \sup_{\phi_j(V_j)} |D^\alpha (f \circ \phi_j^{-1})|.$$

Sobolev Embedding Theorem. *If r is an integer with $r \leq k$ and $r < n/p$, then*

$$W^{k,p}(M) \subset W^{k-r,q}(M)$$

with

$$\frac{1}{q} = \frac{1}{p} - \frac{r}{n}.$$

and

$$\|f\|_{W^{k-r,q}(M)} \leq \|f\|_{W^{k,p}(M)}.$$

If r is an integer with $n/p < r \leq k$, then

$$W^{k,p}(M) \subset C^{k-r}(M),$$

and

$$\|f\|_{C^{k-r}(M)} \leq W^{k,p}(M).$$

Corollary. *$H^k(M) \subset C^{k-[n/2]-1}(M)$, and so if $f \in H^k(M)$ for every k, then $f \in C^\infty(M)$.*

Exercise. *Suppose that X is a (infinite dimensional) vector space with two norms $\| \ \|_1$ and $\| \ \|_2$. Suppose that for every $v \in X$ we have*

$$(*) \qquad \|v\|_1 \ \leq \ \|v\|_2.$$

If X_i is the completion of X in the norm $\| \ \|_i$, then there is a natural inclusion

$$X_2 \ \subset \ X_1,$$

and (*) continues to hold.

Elliptic Regularity Theorem. *Suppose that L is a second order elliptic operator on M (with smooth coefficient). Then*

$$L : C^\infty(M) \ \to \ C^\infty(M)$$

extends to a bounded linear map

$$(1) \qquad L : H^k(M) \ \to \ H^{k-2}(M).$$

(a). If

$$Lu \ = \ f,$$

and $f \in H^{k-2}(M)$, then $u \in H^k(M)$. In particular if $Lu = 0$ then $u \in C^\infty(M)$.
(b). If are no solutions to $Lu = 0$, then L has an inverse $L^{-1} : C^\infty(M) \to C^\infty(M)$ extending to a bounded inverse of the map (1).

(c). If L is self adjoint in the sense that for some smooth density dV on M and all $u, v \in C^\infty(M)$ we have

$$\int_M uLv \, dV \ = \ \int_M (Lu)v \, dV,$$

then if we write $H = \{v : \int uv \, dV = 0$ whenever $Lu = 0\}$, then $L : C^\infty(M) \cap H \to C^\infty(M) \cap H$ has an inverse which extends to a bounded inverse of the map

$$L : H^k(M) \cap H \ \to \ H^{k-2}(M) \cap H.$$

Solution to the problem when $\int K \, dA = 0$. On the closed surface M, we wish to solve

$$-\Delta u \ + \ K \ = \ Ce^{2u},$$

with C a constant. In fact we can get $C = \pm 1$ or $C = 0$. First we deal with the case

$$\int_M K \, dA \ = \ 0.$$

In this case we can solve

$$(2) \qquad -\Delta u \ + \ K \ = \ 0.$$

Indeed, Δ is self-adjoint, and the only solutions to $\Delta u = 0$ are the constants. But we are assuming that K is L^2-orthogonal to the constants, and so there exists a unique u satisfying (2) with $\int_M u \, dA = 0$. The new metric with vanishing curvature is $e^{2u}g$.