**Lecture 1: Uniformization.**

$M$ is a smooth closed surface with metric $g$. The area element is $dA$ and the Gauss curvature is $K$. We are seeking $u \in C^\infty(M)$ so that $e^{2u}g$ has constant curvature. If this curvature is $C$, we wish to solve

$$\Delta u \;-\; K \;=\; -Ce^{2u}.$$

Today we will study the case

$$\int_M K\, dA \;<\; 0.$$

First by rescaling the metric we can assume that the average curvature is $-1$. Indeed, if $\alpha > 0$ is constant and $K_\alpha$ is the curvature for $\alpha^2 g$, then $K_\alpha = \alpha^{-1}K$ and $dA_\alpha = \alpha^2 dA$. Hence

$$\frac{1}{A_\alpha}\int_M K_\alpha\, dA_\alpha \;=\; \frac{1}{\alpha^2}\frac{1}{A}\int_M K\, dA.$$

Hence we start by assuming

$$\frac{1}{A}\int_M K\, dA \;=\; -1,$$

Now we solve

(1) $$\Delta u \;-\; K \;=\; e^{2u},$$

**Uniqueness. (Energy method)**. Suppose that $u, u'$ are two solutions to (1) and consider $u - u'$ which solves

$$\Delta(u' - u) \;=\; e^{2u'} - e^{2u}.$$

Then multiplying by $u' - u$ and integrating over $M$, we have

$$\int_M (u' - u)\Delta(u' - u)\, dA \;=\; \int_M (u' - u)(e^{2u'} - e^{2u})\, dA.$$

However, the left hand side is $\leq 0$ and the right hand side is $\geq 0$, so both sides equal zero.

Now we set $f = K + 1$, so that $f$ has average value 0. We re-write (1):

(2) $$\Delta u \;-\; e^{2u} \;+\; 1 \;=\; f.$$

**Inverse function theorem**. Suppose that $X$ and $Y$ are Banach spaces. We say that

$$f : X \to Y$$

is a differentiable map if for each $x_0 \in X$, there exists a bounded linear map $D_{x_0} f : X \to Y$ such that

$$\sup_{0<\|u\|\leq \varepsilon} \frac{\|f(x_0+u) - f(x_0) - (D_{x_0} f)u\|}{\|u\|} \ \to\ 0, \qquad \text{as } \varepsilon \to 0.$$

We say that $f$ is $C^1$, if it is differentiable and moreover, for each fixed $x_0$,

$$\|D_x f - D_{x_0} f\| \ \to\ 0, \qquad\qquad \text{as } x \to x_0 \text{ in } X,$$

where the norm on the left is the operator norm.

**Theorem.** *If $f : X \to Y$ is $C^1$ and $f(x_0) = y_0$ and $D_{a_0} f$ is a linear isomorphism, then there exists a neighborhood $V$ of $y_0$ in $Y$ and a neighborhood $U$ of $x_0$ such that $f|_U$ maps $U$ onto $V$ and has a $C^1$ inverse.*

**Exercise**. In dimension $n = 2$, if $k \geq 2$ then $H^k(M)$ is a Banach Algebra, *i.e.* there exists $C$ such that

$$\|uv\|_{H^k(M)} \ \leq\ C\|u\|_{H^k(M)}\|v\|_{H^k(M)}.$$

Now we will apply the inverse function theorem to

$$F : H^k \ \to\ H^{k-2},$$

where $k \geq 2$, defined by

$$F(u) \ =\ \Delta u \ -\ e^{2u} \ +\ 1.$$

To find the linearlized operator we look at

$$\begin{aligned} F(u+h) \ -\ F(u) \ =\ \Delta h \ -\ e^{2u} \ +\ e^{2(u+h)} \ &=\ \Delta h \ -\ e^{2u}(e^{2h} - 1) \\ &=\ \left(\Delta \ -\ 2e^{2u}\right) h \ +\ e^{2u}(e^{2h} \ -\ 1 \ -\ 2h). \end{aligned}$$

Because of $H^k$ is a Banach algebra we see that for $u \in H^k(M)$ fixed,

$$D_u F \ := \ \Delta \ -\ 2e^u$$

is a bounded linear map, and for $\|h\| < 1$,

$$\|e^{2u}(e^{2h} - 1 - h)\|_{H^{k-2}(M)} \ \leq\ \|e^{2u}(e^{2h} - 1 - h)\|_{H^k(M)} \ \leq\ C\|e^{2u}\|_{H^k(M)}\|h\|^2_{H^k(M)}.$$

Hence $F$ is differentiable and $D_uF = \Delta - 2e^{2u}$. Moreover, $F$ is $C^1$ because

$$\begin{aligned}\|(D_uF - D_{u+h}F)v\|_{H^{k-2}(M)} &= 2\|(e^{2u} - e^{2(u+h)})v\|_{H^{k-2}(M)} \\ &\leq 2\|e^{2u}\|_{H^k(M)}\|1 - e^{2h}\|_{H^k(M)}\|v\|_{H^k(M)}.\end{aligned}$$

Hence

$$\|(D_uF - D_{u+h}F)\| \leq C\|1 - e^{2h}\|_{H^k(M)} \to 0, \qquad \text{as } h \to 0 \text{ in } X.$$

We need to check that $D_uF = \Delta - 2e^{2u}$ is a linear isomorphism to apply the inverse function theorem. Actually we only stated the elliptic regularity when the coefficients are smooth, but it works fine with $H^k$ coefficients when $H^k$ is a Banach algebra. We need to show that $D_uF$ has trivial kernel. But this holds because

$$\int_M v(\Delta - 2e^{2u})v\,dA = -\int_M \langle \nabla v, \nabla v\rangle\,dA - 2\int_M e^{2u}v^2\,dA \leq 0,$$

with equality if and only if $v = 0$. Here, we use the fact that

$$\int_M v\Delta v\,dA = -\int_M \langle \nabla v, \nabla v\rangle$$

provided $v \in H^2$. This is proved by writing $v$ as a limit of $C^\infty$ functions.

**Continuity**. We consider the set $S \subset [0,1]$ such that $F(u) = tf$ has a solution. Then $0 \in S$ since $F(0) = 0$. The inverse function theorem implies that $S$ is open, since if $F(u_t) = tf$, then there exists an $H^{k-2}(M)$ neighborhood of $tf$ on which the equation has a solution. If we can just prove that $S$ is closed, then we can conclude that $S = [0,1]$ and the original equation has a solution. What we need to show then is the following:

Suppose that $f \in C^\infty(M)$ has average value zero, and $t_n \to t_*$ in $[0,1]$. Suppose that we have a sequence of functions $u_n$ such that $F(u_n) = t_nf$. Then $u_n$ converges to $u$ in $C^\infty(M)$, and $F(u) = tf$.

To prove this we need **A priori bounds**.