MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 7: Uniformization. (See Christodoulou-Klainerman's book on relativity.)

On the closed surface M we are looking at the operator

$$F(u) \;=\; \Delta u \;-\; e^{2u} \;+\; 1.$$

We wish to solve

$$F(u) \;=\; f,$$

where $\int_M f\, dA = 0$. We are using the continuity method. We consider the set $S \subset [0,1]$ such that $F(u) = tf$ has a solution. So far we showed that S is open. Now we need to show that if $t_n \in [0,1]$ and $F(u_n) = t_n f$ with $t_n \to t_*$, then we can find a subsequence $u_{n_i} \to u$ with $F(u) = t_* f$.

Remark. If $f \in C^\infty(M)$ and $u \in H^k$ with $k \geq 2$ satisfies $F(u) = f$, we can write this as

$$\Delta u \;=\; e^{2u} - 1 + f.$$

Since H^k is a Banach algebra, the right hand side is in H^k, and Δ is elliptic, so by elliptic regularity, $u \in H^{k+2}(M)$. Continuing in this way we get $u \in C^\infty(M)$. However, the estimate on the H^n and C^n norms of u depend on having an initial H^k bound on u. This is what we need to obtain.

A priori bounds. Suppose that

$$F(u) \;=\; \Delta u \;-\; e^{2u} \;+\; 1,$$

and $u \in C^2$, but we have no information about the norm of u. We are going to use the **maximum principle** to obtain bounds of the form

$$C_1(f) \;\leq\; u \;\leq\; C_2(f)$$

where $c(f)$ and $C(f)$ are constants depending only on f.

Upper bound

$$\Delta u \;=\; e^{2u} \;-\; 1 \;+\; f.$$

Suppose x_M is the point where u attains its max. Then in normal coordinates about x_M,

$$\Delta u(x_M) \;=\; \frac{\partial^2 u}{\partial x_1^2}(0) \;+\; \ldots \;+\; \frac{\partial^2 u}{\partial x_n^2}(0).$$

However, since x_M is the maximum of u, we see that the right hand side is non-positive. Hence $\Delta u(x_M) \leq 0$, and

$$e^{2u(x_M)} - 1 + f(x_M) \;\leq\; 0.$$

Thus

$$e^{2u} \ \le\ 1-\min f.$$

Hence

$$u \ \le\ \frac{1}{2}\log(1-\min f) \ =:\ C_2(f).$$

Unfortunately we cannot get a lower bound this way. We get rid of f by solving

$$\Delta v = f.$$

It is here that we use the assumption $\int_M f\,dV = 0$. Now our equation becomes

$$\Delta(u-v) \ =\ e^{2u}-1.$$

At a minimum point x_m of $u-v$, the left hand side is non-negative. Hence

$$e^{2u(x_m)} \ -\ 1 \ \ge\ 0,$$

and

$$u(x_m) \ \ge\ 0.$$

Thus

$$u-v \ \ge\ u(x_m)-v(x_m) \ \ge\ -v(x_m),$$

and

$$u \ \ge\ \min v \ -\ \max v \ =:\ C_1(f).$$

Now suppose that $t_n \to t_*$ and

$$F(u_n) \ =\ t_n f.$$

Applying the A Priori bounds we have

$$\|u_n\| \ \le\ C(f)$$

independent of n. Since

$$\Delta u_n \ =\ e^{2u_n} \ -\ 1 \ +\ t_n f,$$

we see that

$$\|\Delta u_n\|_{L^\infty} \ \le\ C_0(f),$$

independent of n, and so

$$\|(\Delta-1)u_n\|_{L^2} \ \le\ C_1(f).$$

Hence

$$\|u_n\|_{H^2(M)} \ \le\ C_2(f),$$

and iterating

$$\|u_n\|_{H^k(M)} \ \le\ C_k(f).$$

independent of n.

If X and Y are Banach spaces, and there is an embedding $X \subset Y$, then we say this embedding is **compact** if whenever x_n is a bounded sequence in X, then there exists a subsequence x_{n_k} convergent in Y.

Kondrakov Compactness Theorem. *The Sobolev embeddings*

$$W^{k,p}(M) \subset W^{k-r,q}(M), \qquad r < n/p, \qquad r \leq k, \qquad 1/q = 1/p - r/n,$$

and

$$W^{k,p}(M) \subset C^{k-r}(M), \qquad n/p < r \leq k,$$

are compact.

We showed that $\|u_n\|_{H^2}$ is uniformly bounded. Hence by the Kondrakov theorem there exists a subsequence which converges strongly in H^1. Using the fact that $\|u_n\|_{H^3}$ is bounded and taking a subsequence of this subsequence which preserves the first two terms, we get a new subsequence which converges strongly in H^2. Continuing this argument and preserving the first k terms at the kth stage, we can construct a subsequence which converges strongly in H^k for every k. This implies by the Sobolev Embedding theorem that it converges in C^∞. But since

$$\Delta u_{n_i} \ = \ e^{2u_{n_i}} \ - \ t_{n_i} f,$$

in the limit we get

$$\Delta u \ = \ e^{2u} \ - \ t_* f.$$

Hence S is open and closed and $1 \in S$, and we get a smooth solution u to $F(u) = f$. If we choose $f = K + 1$ when $(\int K \, dA)/A = -1$, then e^{2u} has curvature -1.