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Spectral Zeta Functions

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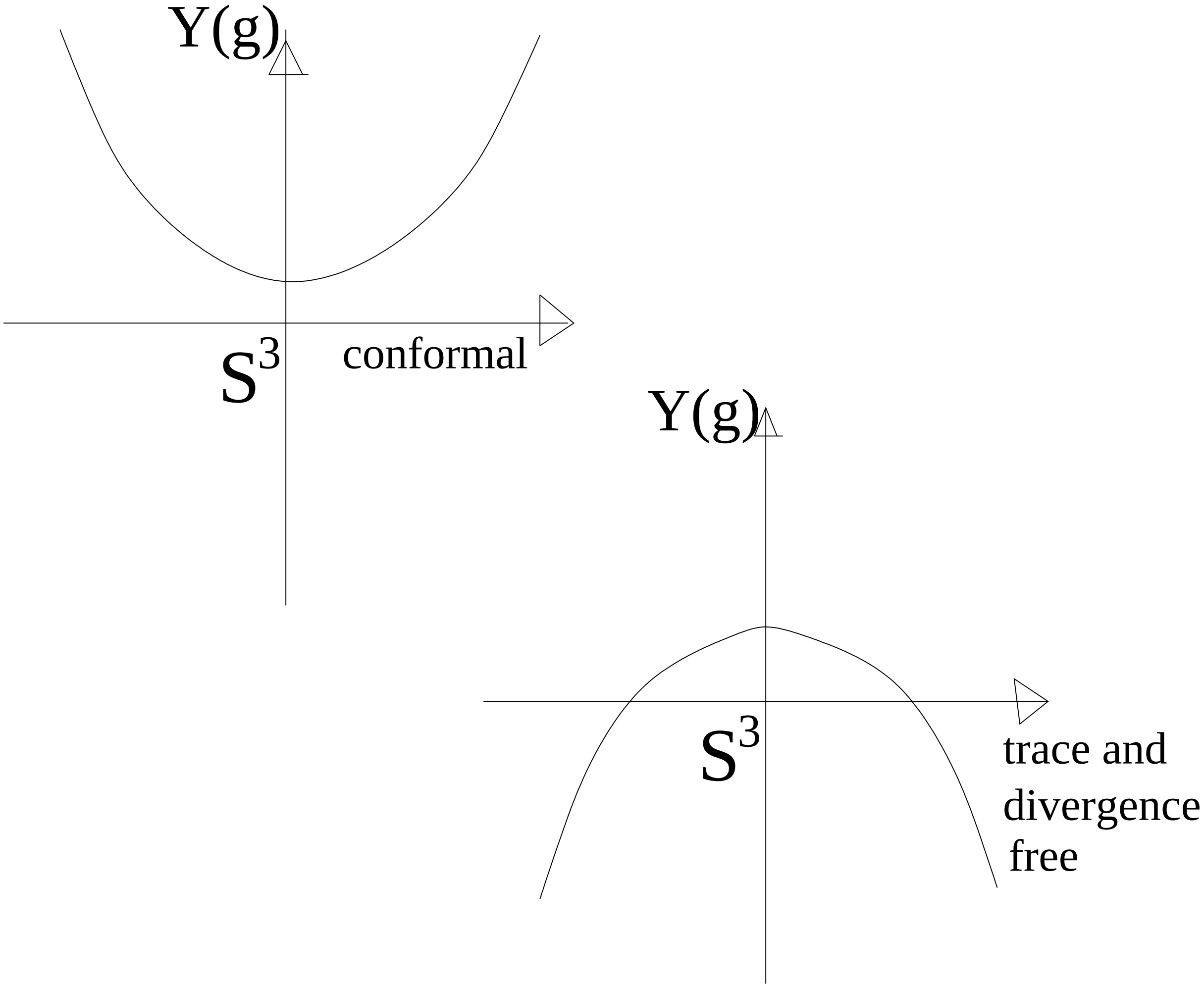
Motivation: $M = M^n$ is a compact, closed, smooth manifold.
g_{ij} is a Riemannian metric on M.
V is the volume associated to g_{ij}.
S is the scalar curvature.

$$Y(g) = \frac{\int_M S\,dV}{V^{(n-2)/n}}.$$

Yamambe Theorem.(Yamabe, Aubin, Yau, Schoen.) Within each conformal class, the minimum value of $Y(g)$ is attained at a metric of constant scalar curvature.

Problem with the Yamabe functional:



Geometric Operators of Laplace Type:

Δ_p is the Hodge Laplacian on p-forms.
(So $\Delta_0 =$ Laplace-Beltrami Operator.)
Δ_C is the conformal Laplacian.

$$\Delta_C = \Delta_0 + \mu S, \qquad\qquad \mu = \frac{n-2}{4(n-1)}$$

$$(\Delta_C)_{e^{2\phi}g} = e^{(-n/2-1)\phi} (\Delta_C)_g \, e^{(n/2-1)\phi}.$$

More generally Δ acts on $C^\infty(T^{q,p}M)$,

$$\Delta = -IV^{2/n} g^{ij}\partial_i\partial_j + C^i\partial_i + C + C^{k\ell}_{ij}(\partial_k\partial_\ell g^{ij}),$$

$C^i = C^i(V, g, \partial g)$, $C = C(V, g, \partial g)$, $C^{k\ell}_{ij} = C^{k\ell}_{ij}(V, g)$.

Spectral Zeta Functions: Δ has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty, \quad \lambda_j \sim C j^{2/n}$.

$$Z_{\Delta}(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s},$$

converges for $\Re s > n/2$. Extends to a meromorphic function with simple poles at

$$\begin{aligned} s &= n/2, n/2-1, \ldots, 1, && n \text{ even}, \quad (1) \\ & n/2, n/2-1, \ldots\ldots\ldots, && n \text{ odd}. \quad (2) \end{aligned}$$

$$\det \Delta = e^{-Z'(0)}.$$

Questions:

1. How do $Z_{\Delta}(s)$, $\det \Delta$ depend on the metric?

2. To what extent can the critical metrics be classified?

3. Do critical metrics exist?

Polyakov-Ray-Singer variation formula, '81

M = 2-dimensional closed compact manifold.

$\tilde{g}$, g = smooth **conformal** metrics on M.

$\tilde{\Delta}$, Δ = Laplace-Beltrami operators for $\tilde{g}$, g.

$$\boxed{\tilde{g} = e^{2\phi} g} \qquad \boxed{\tilde{\Delta} = e^{-2\phi} \Delta}$$

$$\log\det\tilde{\Delta} \; - \; \log\det\Delta$$
$$= \; \frac{-1}{6\pi}\left(\int_M K\phi \, dA \; + \; \frac{1}{2}\int_M |\nabla\phi|^2 \, dA\right) + \; \log\tilde{A} \; - \; \log A.$$

K = Gauss curvature for metric g.

∇ = gradient for g.

dA = area element for g.

$\tilde{A}$, A = areas of M in metrics $\tilde{g}$, g.

For the function det Δ_c**:**

Standard		FIX
S^2	global max	area
S^4	global min	volume & conformal class
S^6	global max	volume & conformal class
S^{4m+3}	local max	nothing
S^{4m+1}	local min	nothing

(Onofri, Branson-Chang-Yang, Branson, -O.)

For the function det **Dirac^2:**

Standard		FIX
S^2	global min	area
S^4	global max	volume & conformal class
S^6	global min	volume & conformal class

(Onofri, Branson-Chang-Yang, Branson.)

For the function $\det \Delta_0$**:**

Standard		FIX
S^2	global max	area
S^3	local max	volume
S^{2n+1} $n > 2$	saddle	volume & conformal class

For the function $Z_{\Delta_c}(s)$**,** $s > n/2$**,** $s \neq n/2-1$**:**

Standard		FIX
S^n	global max	volume & conformal class

(Onofri, Richardson-O, -O, Morpurgo.)

Other critical metrics:
Osgood-Phillips-Sarnak, 1988: If M is a closed surface then of all metrics in a given conformal class and of a given area, the uniform metric maximizes $\det \Delta_0$.

Chang-Yang, 1995: For the following cases of 4-manifolds M, the standard metric is a global minimum for $\det \Delta_c$ among metrics in the same conformal class and with the same volume: S^4, CP^2, $S^2 \times S^2$, R^4/Γ for any lattice Γ, $H^2 \times H^2/\Gamma$ for any lattice Γ, CH^2/Γ for any lattice Γ, $\Sigma \times S^2$ with Σ hyperbolic and those Kähler-Einstein surfaces which are not locally symmetric.

Recent application in four dimensions:

Chern-Gauss-Bonnet: M is a 4-manifold.

$$8\pi^2\chi(M) \;=\; \int |W|^2\, dV \;+\; \int \sigma_2(\Lambda)\, dV,$$

where W is Weyl curvature, Λ is $\mathrm{Ric} - \frac{1}{6}Sg$, σ_2 is the second symmetric function.

Chang-Gursky-Yang, 2000:

If $Y(g) > 0$ and $\int \sigma_2(\Lambda)\, dV > 0$, then there exists a metric conformal to g with $\sigma_2(\Lambda)$ constant, and $\mathrm{Ric} > 0$.

Theorem -O, 2001: Δ is a geometric operator of Laplace type on $C^\infty(T^{q,p}(M))$. Then

$$\text{Hess}\, Z_\Delta(s)(h,h) \;=\; \int_M h((U_s + V_s)h)\, dV.$$

$U_s = U_s(\Delta)$, $V_s = V_s(\Delta)$ are self-adjoint pseudodifferential operators on $C^\infty(S^2(TM))$, meromorphic in s.
U_s is polyhomogeneous of degree $n - 2s$.
V_s is polyhomogeneous of degree 2.
U_s and V_s have poles in $n/2 + Z$.
The symbol expansion of U_s is computable.
The results are preserved under s-differentiation.

The space X of Riemannian metrics on M:
$T_gX = C^\infty(S^2(TM))$.
X can be given the intrinsic L^2 or flat metric. Covariant differentiation exists. $\text{Hess}\, Z_\Delta(s)$ exisits.

$$T_gX = \text{conformal} + \text{trace and divergence free} + \text{diffeomorphic}.$$

At a critical metric, $\text{Hess}\, Z_\Delta(s)$ has the form

form for U_s+V_s on conformal	lower order	0
lower order	form for U_s+V_s on trace&div free	0
0	0	0

Theorem, -O: For the family
$\Delta \;=\; V^{2/n}(\Delta_0 + c_1 \mu S) \;+\; c_2 I \;+\; c_3 \Pi$:

On **trace and divergence free** directions,

$$\sigma(U_s) \;=\; \frac{\Gamma(s - n/2)}{\Gamma(s)} \; f(s) \; I \; |\xi|^{n-2s},$$

On **conformal** directions,

$$\sigma(U_s) \;=\; \frac{\Gamma(s - n/2)}{\Gamma(s)} \; f(s) \; q(s) \; |\xi|^{n-2s},$$

$f(s) > 0$ for $s < n/2$, $\qquad q(s) \;=\; as^2 + bs + c,$

$a = 4(\,(n-2)(c_1 - 1) - 1\,)^2,$
$b = -4(n(n-2)(c_1 - 1) - 1)\,((n-2)(c_1 - 1) - 1),$
$c = (n-1)(n+1)(n-2)^2(c_1 - 1)^2.$

Corollaries: 1. For $n > 2$ even, or n odd and $c_1 \neq 1$, at a critical metric for $\log\det\Delta$, on a space of finite codimension (modulo diffeomorphisms),

$$\text{Hess}\log\det\Delta > 0, \qquad n = 0, 1 \mod 4$$

$$\text{Hess}\log\det\Delta < 0, \qquad n = 2, 3 \mod 4$$

2. If $s < n/2 - 1$ and $q(s) < 0$ then every critical metric for $(d/dz)^k Z_\Delta(s)$ is an essential saddle.

Theorem -O: For this family of Laplacians, one can compute $\sigma(V_s)$ at critical metrics.
For $s > \max\{n/2, 2\}$, the sign of $\operatorname{Hess} Z_\Delta(s)$ on a space of finite codimension is:

Direction	$c_1 \leq 0$	$c_1 = 1$	$c_1 \geq 1$
conformal	> 0	< 0	> 0
trace & div free	< 0	> 0	> 0

So if $c_1 \leq 1$, then every critical metric is an essential saddle.

Theorem, O-Wang: For the Hodge Laplacians Δ_p, on **conformal** directions,

$$\sigma(U_s) = \frac{\Gamma(s-n/2)}{\Gamma(s)} \, f(s) \, Q(s) \, |\xi|^{n-2s},$$

$f(s) > 0$ for $s < n/2$, $\qquad Q(s) = as^2 + bs + c$,

$a = 4(n-2p)^2(1-2/n) + 1$,
$b = -4((n-2p)^2(n-3+1/n) + 2n - 1 - 4p(n-p)/n)$,
$c = (n-1)(n+1)((n-2p)^2(1-4/n)+4)$.

Corollary: For the Hodge Laplacians, $\det \Delta_p$ has the same behavior at critical metrics within the conformal class as $\det \Delta_0$.
(The Dirac operator has different behavior.)

Critical Metrics for $\det \Delta_0$**:**

$G(x,y)$ is **Green's function** for $\det' \Delta$. In normal coordinates $G(x,y) \sim \frac{C}{|x-y|^{n-2}} + \dots$. When n is odd: can canonically subtract off the singular terms to get $G_{\text{reg}}(x,y)$ smooth across the diagonal.

Theorem: When n is odd, g is critical for $\det \Delta_0$

Richardson: under conformal volume preserving deformations $\Leftrightarrow$ $\mathcal{G}_{\text{reg}}(x,x)$ is constant.

-O: Under all volume preserving variations $\Leftrightarrow$ $\mathcal{G}_{\text{reg}}(x,x)$ is constant and

$$(*) \quad \Big(D_{x,i} D_{x,j} \mathcal{G}_{\text{reg}}(x,y) \Big)_{y=x} = \frac{g_{ij}}{n}.$$

Chiu: (*) not satisfied for all flat 3-tori.

Proof that:

$$\text{Hess}\, Z_\Delta(s)(h,h) \;=\; \int_M h((U_s+V_s)h)\, dV.$$

U_s is polyhomogeneous of degree $n-2s$.

V_s is polyhomogeneous of degree 2:

Step 1: Mellin transform

$$Z_\Delta(s) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} \text{ trace } e^{-t\Delta}\, dt.$$

$$\tilde{g} = g + \alpha h.$$

$$Z_\Delta(s)''(\alpha) \;=\; -s \text{ trace } \Delta''\Delta^{-s-1}$$

$$+\frac{1}{\Gamma(s)} \text{ trace } \int_0^\infty\int_0^\infty (u{+}v)^s \Delta' e^{-u\Delta}\Delta' e^{-v\Delta}\, du dv.$$

Step 2: Locally,

$$\Delta' = \sum_{|\beta|+|\gamma|\leq 2} C_{\beta\gamma}(\partial^\beta h)\partial^\gamma.$$

(Similar expression for Δ''.)

The terms $-s$ trace $\Delta''\Delta^{-s-1}$ and

$$\frac{1}{\Gamma(s)} \text{ trace } \int_0^\infty \int_1^\infty (u+v)^s \Delta' e^{-u\Delta} \Delta' e^{-v\Delta}\, dudv$$

contribute to V_s.

We just need to understand

$$\frac{1}{\Gamma(s)} \text{ trace } \int_0^1 \int_0^1 (u+v)^s \Delta' e^{-u\Delta} \Delta' e^{-v\Delta}\, dudv$$

Step 3: This can be written as

$$\frac{1}{\Gamma(s)}\int_0^1\int_0^1 dudv\,(u+v)^s\times$$

$$\int_M\int_M h(x)K_{u,v}(x,y)h(y)\,dV(x)dV(y)$$

where $K_{u,v}$ can be understood using small time asymptotics of the heat kernel.

This equals $\int_M h(W_s h)\,dV$, where W_s has symbol

$$\frac{1}{\Gamma(s)}\int_0^1\int_0^1 (u+v)^s\widehat{K}_{u,v}(x,\xi)dudv.$$

Step 4: Technical Lemma: This integral converges absolutely.

Step 5: Make a double expansion of $\widehat{K}_{u,v}(x,\xi)$. One to extract the symbol expansion of U_s and one to extract the symbol expansion of V_s.

Simple example: R^n, no derivatives: Fourier transform of $(uv)^{-n/2}e^{-|x|^2/(4u)}e^{-|x|^2/(4v)}$ gives $\sigma(T_s)\ =$

$$\frac{1}{\Gamma(s)}\int_0^1\int_0^1 (u+v)^{s-n/2}e^{-|\xi|^2 uv/(u+v)}dudv.$$

Changing variables, this is essentially

$$|\xi|^{n-2s-4}\frac{1}{\Gamma(s)}\int_{T=0}^{|\xi|^2} T^{s-n/2+1}F(T)\,dT,$$

$$\begin{aligned} F(T) \ &\sim \ O(1), T\to 0, && (3)\\ C_1/T + C_2/T^2 &+ \ldots, T\to\infty. && (4)\end{aligned}$$

More direct understanding for n odd, $s \in Z$: Canonical Trace, Kontseivich-Vishik:

$$\mathrm{TR}\ Q \ = \ \Big(\ \mathrm{trace}\ QB^{-s}\Big)_{s=0}$$

where B is any positive elliptic differential operator. Well defined for n odd, on the algebra of *odd class operators*. (Includes differential operators, inverses of elliptic differential operators.)

$$Z(2) \ = \ \mathrm{TR}\ \Delta^{-2}.$$

$\tilde{g}$ is a deformation of g.

$$Z(2)''(0) \ = \ -2\mathrm{TR}\ \Delta''\Delta^{-3}$$

$$+\, 4\mathrm{TR}\ \Delta'\Delta^{-1}\Delta'\Delta^{-3} + 2\mathrm{TR}\ \Delta'\Delta^{-2}\Delta'\Delta^{-2}$$

Simplified example: $\Delta' = \phi\Delta, \quad \Delta'' = \phi^2\Delta.$

TR $\Delta''\Delta^{-3}$ and TR $\Delta'\Delta^{-1}\Delta'\Delta^{-3}$ both equal

TR $\phi^2\Delta^{-2} \;=\; \int_M \phi^2(x) K_{\text{reg}}(\Delta^{-2}, x, x)\, dV(x).$

$$\text{TR } \Delta'\Delta^{-2}\Delta'\Delta^{-2} \;=\; \text{TR } \phi\Delta^{-1}\phi\Delta^{-1}$$

$$= \int_M \int_M \phi(x)(K(\Delta^{-1}, x, y))^2 \phi(y)\, dV(x) dV(y).$$

$$(K(\Delta^{-1}, x, y))^2 \;\sim\; C\,|x-y|^{4-2n}$$

has symbol $C'\,|\xi|^{n-4}$.

Work in progress:

1. With C. Wang: Compute $\sigma(U_s)$ for Δ_p across conformal classes.

2. With S. Zelditch: Global behavior of $\det \Delta_p$ within the conformal class.

3. J. Steiner: Some non-local quantities which arise from critical metrics.

References:

1. Critical metrics for the determinant of the Laplacian in odd dimensions, *Annals of Math. To appear*.

2. Critical metrics for spectral zeta functions, *Preprint*.

3. With C. Wang: Hessian of the zeta function for the Hodge Laplacian, *In preparation.*