SOME FORMULAS FOR THE ENERGY FUNCTIONAL OF A MARKOV PROCESS

by

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1. Introduction

In this paper we shall establish two formulas relating the energy functional of a Markov process to that of a subprocess. Let X be a right Markov process and M an exact multiplicative functional of X. Writing (X, M) for the corresponding subprocess, let L and L^M denote the energy functionals of X and (X, M) respectively. Suppose that M doesn't vanish on $[0, \zeta[$, and define an additive functional A by $dA_t = -dM_t/M_{t-}$. Then given an X-excessive measure ξ and an X-excessive function u we have

$$L^M(\xi, u) = L(\xi, u) + \nu^\xi(u), \tag{1.1}$$

where ν^ξ is the Revuz measure of A relative to X and ξ. Formula (1.1) appears as (3.27) in [**GSt**] in the special case $M_t = e^{-qt}$.

The validity of (1.1) relies heavily on the strict positivity of M. Our second formula serves as a replacement for (1.1) in the general case. To state this formula we need the balayage operator R_M associated with $M; R_M$ operates on the cone of X-excessive

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measures and is the dual of the operator P_M defined by

$$(1.2) \qquad P_M f(x) = \begin{cases} -P^x \int_0^\infty f\circ X_t \, dM_t, & x \in E_M \\ f(x), & x \in E \backslash E_M, \end{cases}$$

where $E_M = \{x \in E\colon P^x(M_0 = 1) = 1\}$. P_M is the analogue of the hitting operator P_B with which it coincides if $M = 1_{[0,T_B[}$, T_B being the hitting time of B. This notation established we have our second formula

$$(1.3) \qquad \begin{aligned} L(\xi, u) &= L^M(\tilde{\xi}, \tilde{u}) + L(\xi, P_M u) \\ &= L^M(\tilde{\xi}, \tilde{u}) + L(R_M \xi, u), \end{aligned}$$

where ξ and u are as for (1.1), $\tilde{\xi} = \xi - R_M\xi$, and $\tilde{u} = u - P_M u$. (See §3 for the precise definition of R_M, $\tilde{\xi}$, and $\tilde{u}$.)

Section 2 contains the proof of (1.1); this proof depends on (2.2) which is of interest in its own right. The balayage operator R_M is discussed in §3; here we omit many details, referring the reader to [**FG**] for full proofs in the special case $M = 1_{[0,T_B[}$. Formula (1.3) is proved in §4. The argument involves a probabilistic identification of the three terms in (1.3) and is perhaps of more interest than the formula itself.

In the remainder of this section we set down our notation and blanket hypotheses. Unexplained terminology can be found in [**Sh**] or [**FG**] and the reader is advised to have a copy of the latter at hand for reference.

Throughout the paper, $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ is a right Markov process in the sense of [**Sh**; §20] with state space $(E, \mathcal{E})$, semigroup (P_t), and resolvent (U^q). In particular, E is a separable Radon space and $\mathcal{E}$ is the σ-field of Borel sets for the Ray topology, $(\mathcal{F}_t)$ is the usual augmentation of the natural filtration of X, and conditions (20.4) and (20.5) in [**Sh**] are in force. The semigroup of X need only be subMarkovian, and ζ denotes the lifetime of X.

Let $S^q(X)$ denote the cone of q-excessive functions of X and put $\mathcal{E}^e = \sigma\{\bigcup_{q\geq 0} S^q(X)\}$. As a rule the letter f always denotes a positive $\mathcal{E}^e$-measurable function on E. The cone of q-excessive measures for X is denoted $\mathrm{Exc}^q(X)$. As usual, when $q = 0$ it vanishes from the notation; in particular $U = U^0$ is the potential kernel for X. The notational scheme for the various subcones of $\mathrm{Exc}\,(X)$ is as in [**FG**] and [**GSt**]. Thus $\mathrm{Pur}\,(X)$ (resp. $\mathrm{Inv}\,(X)$, $\mathrm{Dis}\,(X)$, $\mathrm{Con}\,(X)$) denotes the class of purely excessive (resp. invariant, dissipative, conservative) elements of $\mathrm{Exc}\,(X)$. The analogous classes over a subprocess (X, M) are denoted $S(X, M)$, $\mathrm{Exc}\,(X, M)$, $\mathrm{Pur}\,(X, M)$, etc.

We fix once and for all an exact multiplicative functional (MF), M, of X as specified in [**FG**; (2.1)]. As a matter of convention we assume that $M_t(\omega) = 0$ for all $t \geq \xi(\omega)$ and $\omega \in \Omega$. Define

(1.4) $$S = \inf\{t\colon M_t = 0\};$$

(1.5) $$E_M = \{x \in E\colon P^x(M_0) = 1\} = \{x \in E\colon P^x(S > 0) = 1\}.$$

Then S is a perfect, though not necessarily exact, terminal time and E_M, the set of permanent points of M, is $\mathcal{E}^e$-measurable and serves as the state space of the subprocess (X, M). We write (Q_t) and (V^q) for the semigroup and resolvent of (X, M). For example

$$Q_t f(x) = P^x(f\circ X_t M_t), \quad t \geq 0,\ x \in E.$$

Associated with M is the additive functional (AF) of (X, S)

$$dA_t = -1_{]0,S[}(t)\, dM_t/M_{t-}$$

which has Revuz measure (relative to ξ)

$$\nu^\xi(f) \colon= \uparrow \lim_{t\downarrow 0} t^{-1} P^\xi \int_{]0,t]} f\circ X_s\, dA_s.$$

We write U^q_M for the q-potential operator of A; namely,

$$U^q_M f(x) = -P^x \left(\int_{]0,S[} e^{-qt} f \circ X_t \, dM_t / M_{t-} \right). \tag{1.6}$$

The formula

$$\nu^\xi(f) = \uparrow \lim_{q\to\infty} q\xi U^{q+p}_M f, \quad p \geq 0, \tag{1.7}$$

can be found in [**FG**; (2.15)]. Note that if $M_t = e^{-qt}\, 1_{[0,\zeta[}$, then $\nu^\xi = q \cdot \xi$. As noted by Meyer [**M66**], if $S \geq \zeta$ a.s., then

$$U^q = V^q + U^q_M V^q; \tag{1.8}$$

this formula plays an important role in the sequel.

Finally, recall the energy functional $L\colon \mathrm{Exc}(X) \times S(X) \to [0,\infty]$ determined by

$$L(\xi, u) = \sup \{\mu(u) \colon \mu U \in \mathrm{Exc}\,(X),\ \mu U \leq \xi\},$$

for $\xi \in \mathrm{Exc}\,(X)$, $u \in S(X)$. The reader is referred to [**DM**; XII-39], [**GSt**], or [**FG**] for the various properties of the bilinear form L. The energy functional of (X, M) is denoted L^M, and that of the q-subprocess $X^q = (X, e^{-q\cdot}\, 1_{[0,\zeta[})$ is denoted L^q.

2. Proof of (1.1).

In this section we assume that the MF (M_t) satisfies $M_t > 0$ for all $0 \leq t < \zeta$ a.s.; that is, the terminal time S defined by (1.4) satisfies $P^x(S < \zeta) = 0$ for all $x \in E$. In particular, $E_M = E$.

We begin the proof of (1.1) by noting that it suffices to consider the two special cases (i) $\xi \in \mathrm{Dis}\,(X)$, and (ii) $\xi \in \mathrm{Con}\,(X)$. This is because all three terms in (1.1) are additive in ξ, and since each $\xi \in \mathrm{Exc}\,(X)$ admits a unique decomposition $\xi = \xi_d + \xi_c$ where $\xi_d \in \mathrm{Dis}\,(X)$, $\xi_c \in \mathrm{Con}\,(X)$.

(i) Assume $\xi \in \mathrm{Dis}\,(X)$ and choose a sequence (μ_n) of measures on E such that $\mu_n U \uparrow \xi$. Define $\nu_n = \mu_n + \mu_n U_M$, so that $\nu_n V = \mu_n U \uparrow \xi$ by (1.8) with $q = 0$. By well-known properties of the energy functionals L^M and L (see [**GSt**] or [**FG**]) we have

$$L^M(\xi, u) = \uparrow \lim \nu_n(u) = \uparrow \lim\,[\mu_n(u) + \mu_n U_M(u)]$$

$$= L(\xi, u) + \nu^\xi(u).$$

The last equality above follows from [**FG**; (2.18)]. (This argument is a trivial modification of that used in [**GSt**] for the special case $M_t = e^{-qt} 1_{[0,\zeta[}$.)

Before proceeding with the proof of (1.1) in the case $\xi \in \mathrm{Con}\,(X)$ we record two facts

(2.1) LEMMA. *Given $\xi \in \mathrm{Con}\,(X)$ let $\xi = \xi_i + \xi_p$ be the decomposition of ξ into its invariant and purely excessive parts relative to (X, M). Then both ξ_i and ξ_p lie in $\mathrm{Con}\,(X)$.*

Proof. If $\xi(f) < \infty$ then since $\mathrm{Con}\,(X) \subset \mathrm{Inv}\,(X)$,

$$\xi_p P_t(f) = \xi P_t f - \xi_i P_t f \le \xi(f) - \xi_i Q_t f$$

$$= \xi(f) - \xi_i(f) = \xi_p(f),$$

and so $\xi_p \in \mathrm{Exc}\,(X)$. But then $\xi_p \in \mathrm{Con}\,(X)$ since $\xi_p \le \xi \in \mathrm{Con}\,(X)$. Therefore, $\xi_i = \xi - \xi_p$ is seen first to lie in $\mathrm{Inv}(X)$, then in $\mathrm{Con}\,(X)$, being dominated by ξ. □

Recall that (V^q) is the resolvent of (X, M) and that $V = V^0$.

(2.2) PROPOSITION. *(a) If $\xi \in \mathrm{Con}\,(X) \cap \mathrm{Inv}\,(X, M)$, then $\xi \in \mathrm{Con}\,(X, M)$, $\nu^\xi = 0$, and $M = 1_{[0,\zeta[}$ a.s. P^ξ.*

(b) If $\xi \in \mathrm{Inv}\,(X)$, then $\xi_p = \nu^\xi V$, where ξ_p denotes the purely excessive part of ξ relative to (X, M).

Before proving (2.2) let us use it and (2.1) to finish the proof of (1.1). Thus suppose that $\xi \in \mathrm{Con}\,(X)$. By Lemma (2.1) we need only consider the special cases $\xi \in \mathrm{Con}\,(X) \cap \mathrm{Inv}\,(X, M)$ and $\xi \in \mathrm{Con}\,(X) \cap \mathrm{Pur}\,(X, M)$. In the first case all terms in (1.1) vanish because of (2.2a) and [**GSt**; (3.11)]. In the second case $\xi = \nu^\xi V$ by (2.2b), hence $L^M(\xi, u) = \nu^\xi(u)$ while $L(\xi, u) = 0$ by the properties of L^M and L; see [**FG**] or [**GSt**]. ∎

Proof of (2.2). (a) Assume $\xi \in \mathrm{Con}\,(X) \cap \mathrm{Inv}\,(X, M)$. Then by (1.8), if $q > 0$,

$$\xi = q\xi U^q = q\xi V^q + q\xi U_M^q V^q = \xi + q\xi U_M^q V^q, \tag{2.4}$$

so $\xi U_M^q V^q = 0$ and $\xi U^q = \xi V^q$, even when $q = 0$. Since $U^q \geq V^q$, the last equality yields $U^q(x, \cdot) = V^q(x, \cdot)$ for ξ a.e. $x \in E$. But $\xi \in \mathrm{Con}\,(X)$ so if $f > 0$ then $Vf = Uf = \infty$ a.e. ξ, hence $\xi \in \mathrm{Con}\,(X, M)$. Moreover, $U^1 1 = V^1 1$ a.e. ξ, which implies that $M = 1_{[0,\zeta[}$ a.s. P^ξ. Since $V^q 1 > 0$ (recall that $E_M = E$), $\xi U_M^q V^q = 0$ implies $\xi U_M^q = 0$, hence $\nu^\xi = 0$ in view of (1.7).

(b) Now assume $\xi \in \mathrm{Inv}\,(X)$ and fix $f \in bp\mathcal{E} \cap L^1(\xi)$. Since $\xi \in \mathrm{Inv}\,(X) \subset \mathrm{Exc}\,(X, M)$, the first two equalities in (2.4) are valid; letting $q \to \infty$ we obtain

$$\lim_{q \to \infty} q\xi U_M^q V^q f = 0. \tag{2.5}$$

But by (1.7) and the resolvent equation for (V^q)

$$\nu^\xi V f = \uparrow \lim_{p \downarrow 0} \uparrow \lim_{q \uparrow \infty} q\xi U_M^q V^p f = \lim_{p \downarrow 0} \lim_{q \uparrow \infty} q\xi U_M^q (V^q f + (q - p) V^q V^p f).$$

Using now (2.5), (1.8), $\xi \in \mathrm{Inv}\,(X)$, and the resolvent equation, the last displayed line equals

$$\lim_{p \downarrow 0} \lim_{q \uparrow \infty} q(q - p)\xi(U^q - V^q)V^p f = \lim_{p \downarrow 0} \lim_{q \uparrow \infty} q\xi V^q f - p\xi V^p f = \xi_p(f),$$

because $p\xi V^p \to \xi_i$ as $p \to 0$. Thus $\nu^\xi V = \xi_p$ and the proof of (2.2) is complete. ∎

Remark. Suppose that $\xi \in \mathrm{Con}\,(X)$ is minimal; i.e., ξ admits no nontrivial decomposition into a sum of elements of $\mathrm{Exc}\,(X)$. Then by (2.1) and (2.2) either $\xi \in \mathrm{Inv}\,(X, M)$ in which case $M = 1_{[0,\zeta[}$ a.s. P^ξ, or $\xi = \nu^\xi V \in \mathrm{Pot}\,(X, M)$.

3. Balayage.

As in previous sections M is an exact MF of X, $S = \inf\{t: M_t = 0\}$, and $E_M = \{x \in E: P^x(M_0) = 1\}$. However we no longer assume that $S \geq \zeta$ a.s.

For $q \geq 0$ we define

$$
\text{(3.1)} \qquad P_M^q f(x) = \begin{cases} -P^x \int_{]0,\zeta[} e^{-qt} f \circ X_t \, dM_t, & x \in E_M; \\ f(x), & x \in E_M^c, \end{cases}
$$

which agrees with (1.2) when $q = 0$. Clearly $P_M^q(b\mathcal{E}^u) \subset b\mathcal{E}^u$ and $P_M^q(S^q(X)) \subset S^q(X)$. (Here and in the sequel, if $(F, \mathcal{F})$ is a measurable space, then $\mathcal{F}^u$ is the universal completion of $\mathcal{F}$.) As a replacement for (1.8) (which is valid only when $S \geq \zeta$), we have

$$
\text{(3.2)} \qquad U^q = V^q + P_M^q U^q.
$$

The operators P_M^q have duals relative to L^q which we now define following [**FG**; §3] (where the special case $M = 1_{[0,T_B[}$ was considered). First, if $q > 0$ and $\xi \in \mathrm{Exc}^q(X)$,

$$
\text{(3.3)} \qquad R_M^q \xi(f) := L^q(\xi, P_M^q U^q f)
$$

where L^q is the energy functional of the q-subprocess X^q. As in [**FG**], $R_M^q \xi \in \mathrm{Exc}^q(X)$, $R_M^q \xi \leq \xi$ and if $\mu_n U^q \uparrow \xi \in \mathrm{Exc}^q(X)$ then $R_M^q(\mu_n U) = \mu_n P_M^q U^q \uparrow R_M^q \xi$. Next, a straightforward computation shows that

$$
\text{(3.4)} \qquad (q-r) U^r P_M^q U^q \leq P_M^r U^r - P_M^q U^q, \quad 0 < r < q.
$$

Now if $\xi \in \mathrm{Exc}^r(X)$ and $0 < r < q$, then we can choose a sequence of measures μ_n such that $\mu_n U^r \uparrow \xi$; setting $\nu_n = \mu_n + (q-r)\mu_n U^r$ we have $\nu_n U^q \uparrow \xi$, hence by (3.4),

$$
R_M^q \xi = \lim_n \mu_n (I + (q-r)U^r) P_M^q U^q \leq \lim_n \mu_n P_M^r U^r = R_M^r \xi.
$$

Thus $q \mapsto R_M^q \xi$ is decreasing on $]0, \infty[$ if $\xi \in \mathrm{Exc}\,(X)$. We now define

$$
\text{(3.5)} \qquad R_M^0 \xi = R_M \xi = \uparrow \lim_{q \downarrow 0} R_M^q \xi, \quad \xi \in \mathrm{Exc}\,(X).
$$

Evidently $R_M\colon \mathrm{Exc}\,(X) \to \mathrm{Exc}\,(X)$. Various properties of R^q_M, $q \ge 0$, are established in [**FG**; §3] in the special case $M = 1_{[0,T_B[}$. These assertions remain valid here and will be used in the sequel. We mention here a few specifics. First, the relation (3.3) is true when $q = 0$ provided $\xi \in \mathrm{Dis}\,(X)$. Second, the duality formula

$$L^q(R^q_M \xi, u) = L^q(\xi, P^q_M u), \tag{3.6}$$

is valid for $q \ge 0$, $\xi \in \mathrm{Exc}^q(X)$, $u \in S^q(X)$. Finally, writing $\xi = \xi_d + \xi_c$ for the decomposition of $\xi \in \mathrm{Exc}\,(X)$ into dissipative and conservative components, we have $(R_M\xi)_d = R_M(\xi_d)$ and $(R_M\xi)_c = R_M(\xi_c)$.

In the remainder of this section we give a precise definition of $\tilde\xi$ and $\tilde u$ appearing in formula (1.3). In fact we also consider the analogous objects $\tilde\xi^q$ and $\tilde u^q$, $q > 0$.

(3.7) PROPOSITION. *Fix $q \ge 0$. Given $\xi \in \mathrm{Exc}^q(X)$, a σ-finite measure $\tilde\xi^q$ on $(E, \mathcal{E})$ is uniquely determined by*

$$\tilde\xi^q(f) = \xi(f) - R^q_M \xi(f), \quad f \in pL^1(\xi).$$

The measure $\tilde\xi^q$ is carried by E_M, and if regarded as a measure on $(E_M, \mathcal{E} \cap E_M)$ is an element of $\mathrm{Exc}^q(X, M)$. If $\xi = \mu U^q$, then $\tilde\xi^q = \mu V^q$.

Proof. It suffices to consider the case $q > 0$. Then $\tilde\xi^q(E^c_M) = 0$ follows easily from (3.2) and (3.3), as does the evaluation $\tilde\xi^q = \mu V^q$ if $\xi = \mu U^q$. For a general $\xi \in \mathrm{Exc}^q(X)$ ($= \mathrm{Dis}^q(X)$ since $q > 0$), there is a sequence (μ_n) with $\mu_n U^q \uparrow \xi$. Then $\mu_n P^q_M U^q \uparrow R^q_M \xi$ as noted earlier; hence by (3.2), $\mu_n V^q f \to \tilde\xi^q(f)$ provided $f \in pL^1(\xi)$. Finally, for such f,

$$\tilde\xi^q Q^q_t f = \lim_n \mu_n V^q Q^q_t f \le \lim_n \mu_n V^q f = \tilde\xi^q(f),$$

and so $\tilde\xi^q \in \mathrm{Exc}^q(X, M)$. ▯

The dual definition of $\tilde{u}^q$ is a more delicate matter.

(3.8) PROPOSITION. *Fix $q \geq 0$. Given $u \in S^q(X)$, there exists $\tilde{u}^q \in S^q(X, M)$ such that*

$$\tilde{u}^q = u - P_M^q u \quad \text{on} \quad \{P_M^q u < \infty\} \cap E_M.$$

If $u = U^q f$ then $\tilde{u}^q = V^q f$ on $\{P_M^q u < \infty\} \cap E_M$.

Proof. Suppose first that $u \in S^q(X)$ where $q > 0$. Let $B = \{P_M^q u < \infty\} \cap E_M$. Then $B \in \mathcal{E}^e$ is finely open. Since $\{P_M^q u < \infty\}$ is absorbing for X^q, each of the measures $Q_t^q(x, \cdot)$, $t \geq 0$, $x \in B$, is carried by B. Define u^* on E_M by

$$u^*(x) = \begin{cases} u(x) - P_M^q u(x), & x \in B; \\ \infty, & x \in E_M \backslash B. \end{cases}$$

We claim that u^* is (Q_t^q)-supermedian. Indeed since $q > 0$ there are functions f_n such that $U^q f_n \uparrow u$ as $n \to \infty$. Then $V^q f_n = U^q f_n - P_M^q U^q f_n \to u^*$ on B. Fatou's lemma now shows that $Q_t^q u^* \leq u^*$, $t \geq 0$, as claimed. Now define

$$\tilde{u}^q(x) = \uparrow \lim_{t \downarrow 0} Q_t^q u^*(x), \quad x \in E_M.$$

Clearly $\tilde{u}^q \in S^q(X, M)$. Writing $u_n^* = u^* \wedge n$ we see that each u_n^* is (Q_t^q)-supermedian and finely continuous on B. Thus if $x \in B$,

$$Q_t^q u_n^*(x) = P^x(u_n^*(X_t) e^{-qt} M_t) \to u_n^*(x), \quad \text{as} \quad t \downarrow 0. \tag{3.9}$$

But $u_n^* \uparrow u^*$ as $n \to \infty$, and $Q_t^q u_n^* \uparrow u_n^*$ on B as $t \downarrow 0$ because of (3.9). It follows that $\tilde{u}^q = u^*$ on B. In particular $\tilde{u}^q = u^* = V^q f$ on B if $u = U^q f$.

It remains to consider the case $q = 0$. Fix $u \in S(X) \subset S^q(X)$ and define $\tilde{u}^q$ as above. Note that $\{P_M u < \infty\} \subset \bigcap_{q > 0} \{P_M^q < \infty\}$. Thus if $x \in \{P_M u < \infty\} \cap E_M$,

$$u^*(x) := u(x) - P_M u(x) = u(x) - \lim_{q \downarrow 0} P_M^q u(x)$$

(we put $u^* = \infty$ on $E_M \backslash \{P_M u < \infty\}$). Letting $q \downarrow 0$ in the inequality $Q_t^q \tilde{u}^q \le \tilde{u}^q$ we see that u^* is (Q_t)-supermedian. Setting $\tilde{u} = \uparrow \lim_{t \downarrow 0} Q_t u^*$, the truncation argument used earlier shows that $\tilde{u} = u^*$ on $\{P_M u < \infty\} \cap E_M$. Moreover $\tilde{u} = Vf$ on $\{P_M u < \infty\} \cap E_M$ if $u = Uf$. ∎

Remark. It is easy to check that the mappings $\xi \mapsto \tilde{\xi}^q$ and $u \mapsto \tilde{u}^q$ are "positive linear" on their respective domains $\mathrm{Exc}^q(X)$ and $S^q(X)$.

4. The Second Formula

We are now ready to give a precise statement, and proof, of formula (1.3). We shall first state and prove the result for a *Borel* right process X; i.e., E is a Lusin metrizable space with Borel sets $\mathcal{E}$, and each P_t maps $b\mathcal{E}$ into itself. This will enable us to use the Kuznetsov process associated with X and $\xi \in \mathrm{Exc}\,(X)$, and also Meyer's perfection theorem [**M74**] for M. To the best of our knowledge neither the existence of the Kuznetsov process nor Meyer's theorem have been established for general right processes. In (4.19) we shall indicate how formula (3.1) for general right processes may be reduced to the Borel case. As in previous sections M is an exact MF of X. We maintain the notation established in previous sections.

(4.1) THEOREM. *Assume that X is a Borel right process. Given $\xi \in \mathrm{Exc}\,(X)$ and $u \in S(X)$,*

$$L(\xi, u) = L^M(\tilde{\xi}, \tilde{u}) + L(\xi, P_M u) = L^M(\tilde{\xi}, \tilde{u}) + L(R_M \xi, u), \tag{4.2}$$

where $\tilde{\xi} \in \mathrm{Exc}\,(X, M)$ and $\tilde{u} \in S(X, M)$ are as defined in §3.

Of course, the second equality in (4.2) expresses the duality between P_M and R_M already noted. We leave it to the reader to check that these formulas are trivial if either $\xi = \mu U$ or $u = Uf$ provided the obvious finiteness conditions are satisfied. However we have been unable to use this fact to extend (4.2) to $\xi \in \mathrm{Dis}\,(X)$ in any straightforward manner. The problem is that $\xi \mapsto \tilde{\xi}$ is *not* monotone. .

Henceforth we assume that X is a Borel right process; see [**G75**] or [**Sh**; (20.6)]. Our notation for the Kuznetsov process (Y, Q_ξ) associated with X and $\xi \in \mathrm{Exc}\,(X)$ is that used in [**F**] and differs slightly from that used in [**FM**] or [**GSt**]. Let W denote the space of paths $w : \mathbb{R} \to E \cup \{\Delta\}$ that are E-valued and right continuous on an open interval $]\alpha(w), \beta(w)[$ and that take the value Δ elsewhere. The dead path $[\Delta] : t \to \Delta$ corresponds to $]\alpha, \beta[= \emptyset$, and the appropriate convention is $\alpha([\Delta]) = +\infty$, $\beta([\Delta]) = -\infty$. Let $Y = (Y_t : t \in \mathbb{R})$ denote the coordinate process on W, and put $\mathcal{G}^0 = \sigma\{Y_t : t \in \mathbb{R}\}$ and $\mathcal{G}^0_t = \sigma\{Y_s : s \leq t\}$.

Given $\xi \in \mathrm{Exc}\,(X)$, the associated *Kuznetsov measure* Q_ξ is the unique measure on $(W, \mathcal{G}^0)$ not charging $\{[\Delta]\}$ such that if $t_1 < \cdots < t_n$,

$$Q_\xi(\alpha < t_1, Y_{t_1} \in dx_1, \ldots, Y_{t_n} \in dx_n, t_n < \beta) \\ = \xi(dx_1) P_{t_2 - t_1}(dx_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n). \tag{4.3}$$

Evidently Q_ξ is σ-finite and *invariant* relative to the shift operators (σ_t) defined by $Y_t \circ \sigma_s = Y_{t+s}$, $s, t \in \mathbb{R}$. Also defined on W are birthing, killing, and shift operators:

$$\begin{aligned} (b_t w)(s) &= w(s) \quad , \quad s > t; \\ &= \Delta \quad , \quad s \leq t; \\ (k_t w)(s) &= w(s) \quad , \quad s < t; \\ &= \Delta \quad , \quad s \geq t. \end{aligned}$$

$$\theta_t = b_0 \sigma_t = \sigma_t b_t, \quad t \in \mathbb{R}.$$

A convenient way to express the relationship between X and Y is to realize X on the "canonical" space $\Omega = \{w \in W : \alpha(w) = 0,\ Y_{\alpha+}(w) \text{ exists in } E\} \cup \{[\Delta]\}$. We let X_t, $\mathcal{F}^0_t$, and $\mathcal{F}^0$ denote the respective restrictions to Ω of Y_t, $\mathcal{G}_t^0$ and $\mathcal{G}^0$ if $t > 0$ and if $t = 0$ we let $X_0 = Y_{0+}|_\Omega$, $\mathcal{F}^0 = \mathcal{G}_{0+}^0|_\Omega$.

Clearly $X_{t+s} = X_s \circ \theta_t$ if $s, t \geq 0$, while $X_s \circ \theta_t = Y_{s+t}$ if $s \geq 0$, $\alpha < t < \infty$. See [**FM**], [**GSt**], [**G88**] and [**F**] for further details. Since X is Borel, a result of Meyer [**M74**] allows us to assume that the exact MF M is adapted to $(\mathcal{F}_{t+}^u)$. Following [**G88**] we now "extend" M to W: for $\alpha(w) < s \leq t$ define

$$\text{(4.4)} \qquad N(s,t) = \begin{cases} M_{t-s} \circ \theta_s, & \alpha < s < t; \\ 1, & \alpha < s = t. \end{cases}$$

This makes sense since $\theta_s : \{\alpha < s\} \to \Omega$. The map $s \mapsto N(s,t)$ is increasing and right continuous on $]\alpha, t[$. This allows us to define

$$N_t = \begin{cases} \downarrow \lim_{s \downarrow \alpha} N(s,t), & \alpha < t; \\ 1, & \alpha \geq t. \end{cases}$$

Clearly $t \mapsto N_t$ is decreasing and right continuous on $]\alpha, \infty[$, $N_t \circ \sigma_s = N_{t+s}$ for all $s, t \in \mathbb{R}$, and (N_t) is adapted to $(\mathcal{G}_{t+}^u)$. It follows from the multiplicative property of M that $N_s N(s,t) = N_t$ if $\alpha < s < t$. Letting $s \downarrow \alpha$, then $t \downarrow \alpha$, it follows that $N_{\alpha+}^2 = N_{\alpha+}$, whence

$$\text{(4.5)} \qquad N_{\alpha+} = 0 \quad \text{or} \quad 1.$$

See [**G88**] for a complete discussion of these functionals.

The following evaluation is the key to Theorem (4.1).

(4.6) PROPOSITION. *If* $\xi \in \mathrm{Exc}\,(X)$, *then*

$$\text{(4.7)} \qquad R_M \xi(f) = Q_\xi(f \circ Y_t (1 - N_t)), \quad \forall t \in \mathbb{R}.$$

Remarks. The R.H.S. of (4.7) is independent of $t \in \mathbb{R}$ owing to the (σ_t)-invariance of Q_ξ and the homogeneity of N_t. In [**FM**] the formula (4.7) was taken as the *definition* of R_M in the special case $M = 1_{[0,T_B[}$. The identification of this definition with that given in §3 of this paper was made in [**FG**].

Proof of (4.6). First suppose that $\xi \in \mathrm{Dis}\,(X)$. Define H on Ω by $H = \int\limits_0^\infty f{\circ}X_t\,(1 - M_t)\,dt$ and note that H is $\mathcal{F}^u$-measurable and *excessive* in the sense that $s \mapsto H{\circ}\theta_s$ is decreasing and right continuous on $[0,\infty[$. Note that $P^{\cdot}(H) = P_M Uf$. Define H^* on W by

$$H^* = \uparrow \lim_{s \downarrow \alpha} H{\circ}\theta_s.$$

It is shown in [**F**; (2.7)] that since $\xi \in \mathrm{Dis}$ there is a $\mathcal{G}^0$-measurable random time $S^*\colon W \to [-\infty, +\infty]$ such that $t + S^*{\circ}\sigma_t = S^*$ and $Q_\xi(S^* \notin \mathbb{R}) = 0$. Moreover, using [**F**; (4.4)],

$$R_M\xi(f) = L(\xi, P_M Uf) = Q_\xi(H^*; 0 < S^* < 1). \tag{4.8}$$

(The first equality above is just (3.6).) But it is easy to check that

$$H^* = \int_{\mathbb{R}} f{\circ}Y_t(1 - N_t)\,dt = \int dt\,[f{\circ}Y_0(1 - N_0)]{\circ}\sigma_t$$

so the third term in (4.8) is precisely $Q_\xi(f{\circ}Y_0(1 - N_0))$ by the "switching identity" (2.1) in [**F**]. This proves (4.7) in case $\xi \in \mathrm{Dis}\,(X)$.

To handle $\xi \in \mathrm{Con}\,(X)$ we use the fact that $\mathrm{Exc}^q(X) = \mathrm{Dis}^q(X)$ if $q > 0$. Fix $\xi \in \mathrm{Con}\,(X)$ and let Q^q_ξ be the Kuznetsov process for X^q and ξ. Then as a special case of the results in [**G88**], if $F \in p\mathcal{G}^u$ with $F([\Delta]) = 0$,

$$Q^q_\xi(F) = Q_\xi \iint q^2 e^{-q(s-r)} 1_{\{r<s\}} F{\circ}k_s{\circ}b_r\,dr\,ds. \tag{4.9}$$

(One can verify (4.9) directly by checking finite dimensional distributions.) Now since $\xi \in \mathrm{Con}\,(X) \subset \mathrm{Dis}^q(X)$, by what has already been proved,

$$R^q_M\xi(f) = Q^q_\xi(f{\circ}Y_0(1 - N_0)) = Q_\xi(f{\circ}Y_0 \int_{-\infty}^0 q\,e^{qr}(1 - N_0{\circ}b_r)\,dr),$$

since $N_0 \circ k_s = N_0$ if $\alpha < 0 < s$. If $r \le \alpha$ then $N_0 \circ b_r = N_0$, while if $\alpha < r < 0$ then $N_0 \circ b_r = M_{-r} \circ \theta_r$. Therefore

$$
\begin{aligned}
(4.10) \qquad R_M^q \xi(f) &= Q_\xi(f \circ Y_0 (1 - N_0) e^{q\alpha}) \\
&\quad + Q_\xi(f \circ Y_0 \int_\alpha^0 q e^{qr} (1 - M_{-r} \circ \theta_r)\, dr).
\end{aligned}
$$

But $r \mapsto M_{-r} \circ \theta_r = N(r, 0)$ is increasing and right continuous on $]\alpha, 0[$ with limit N_0 as $r \downarrow \alpha$. Integrating by parts in (4.10) we obtain

$$
R_M^q \xi(f) = Q_\xi(f \circ Y_0 \int_{]\alpha,0]} e^{qr}\, d_r N(r, 0)).
$$

Consequently $R_M \xi(f) = \uparrow \lim_{q \downarrow 0} R_M^q \xi(f) = Q_\xi(f \circ Y_0 (1 - N_0))$, since $N(0,0) = 1$. The proof of Proposition (4.6) is complete. ☐

For the proof of (4.2) in case $\xi \in \mathrm{Con}\,(X)$ we need the following extension of [**GSt**; (6.12)].

(4.11) COROLLARY. *Define $\varphi = 1 - P^{\cdot}(M_{\zeta-})$. Then $\varphi \in S(X)$, and if $\xi \in \mathrm{Con}\,(X)$ then $R_M \xi = \varphi \cdot \xi$.*

Proof. The fact that $\varphi \in S(X)$ is a consequence of the exactness of M which implies that $M_{\zeta-} \circ \theta_t$ decreases to $M_{\zeta-}$ as t decreases to 0. Fix $\xi \in \mathrm{Con}\,(X)$ so that $Q_\xi(\alpha > -\infty) = Q_\xi(\beta < +\infty) = 0$. Recall that (N_t) is decreasing and homogeneous: $N_t \circ \sigma_s = N_{t+s}$. Thus if $S_\lambda = \inf\{t > \alpha : N_t < \lambda\}$ then $t + S_\lambda \circ \sigma_t = S_\lambda$ for all $t \in \mathbb{R}$. Accordingly, by [**F**; (2.7)], since $\xi \in \mathrm{Con}\,(X)$ we must have $Q_\xi(S_\lambda \in \mathbb{R}) = 0$ for all $\lambda \in]0,1[$. But $N_{\alpha+} = 0$ or 1 by (4.5); since $t \mapsto N_t$ is decreasing and $[0,1]$-valued, it follows that $N_t = N_{\alpha+}$ for all $t > \alpha$, a.s. Q_ξ. Consider now the "dual" $(\hat{N}_t)$ of (N_t) defined by

$$
\hat{N}_t = \begin{cases} \uparrow \lim_{u \uparrow \beta} N(t, u) = M_{\zeta-} \circ \theta_t, & \alpha < t < \beta; \\ 0, & t \le \alpha; \\ 1, & t \ge \beta. \end{cases}
$$

Clearly $(\hat{N}_t)$ is increasing and homogeneous. Reasoning as for (N_t) we see that $\hat{N}_t = \hat{N}_{\beta-}$ for all $t < \beta$ a.s. Q_ξ. But $N(s,u) = N(s,t)N(t,u)$ for $\alpha < s < t < u < \beta$, so

$$N_{\beta-} = N_t \hat{N}_t = \hat{N}_{\alpha+}, \quad \alpha < t < \beta.$$

This and the fact that $Q_\xi(\alpha > -\infty) = Q_\xi(\beta < +\infty) = 0$ yields

$$N. \equiv \hat{N}. \equiv 0 \text{ or } 1 \text{ a.s. } Q_\xi.$$

But $\hat{N}_0 = M_{\zeta-}\circ\theta_0$ on $\{\alpha < 0 < \beta\}$, so using (4.7),

$$\begin{aligned} R_M\xi(f) &= Q_\xi(f\circ Y_0(1 - N_0)) = Q_\xi(f\circ Y_0(1 - \hat{N}_0)) \\ &= Q_\xi(f\circ Y_0(1 - M_{\zeta-}\circ\theta_0)) = Q_\xi(f\circ Y_0\, \varphi\circ Y_0) = \xi(\varphi \cdot f) \end{aligned}$$

as claimed. □

Proof of (4.2).

(i) $\xi \in \mathrm{Con}\,(X)$: Since $R_M\xi \in \mathrm{Con}\,(X)$ if $\xi \in \mathrm{Con}\,(X)$, it suffices to show that $\tilde{\xi} \in \mathrm{Con}\,(X, M)$, for then all terms in (4.2) will vanish. By (4.11), $\tilde{\xi} = (1-\varphi)\cdot\xi$. Since $P^\xi(\zeta < \infty) = 0$, if we write $M_\infty = \lim_{t\uparrow\infty} M_t$, then for $t \geq 0$, and ξ-a.e. $x \in E$,

$$\begin{aligned} \varphi(x) &= P^x(1 - M_t) + P^x(M_t(1 - M_\infty\circ\theta_t)) \\ &= P^x(1 - M_t) + P^x(M_t\varphi\circ X_t) = P^x(1 - M_t) + \varphi(x)P^x(M_t), \end{aligned}$$

where the last equality follows since $t \mapsto \varphi\circ X_t$ is constant a.s. P^ξ (see [**GSt**; (2.9)]). Letting $t \to \infty$ above we arrive at $\varphi = \varphi^2$ a.e. ξ. Thus $\tilde{\xi} = 1_{\{\varphi=0\}}\cdot\xi$. Choose $f \in pL^1(\xi)$ such that $\{f > 0\} = E_M$. Then $\xi\{0 < Uf < \infty\} = 0$ since $\xi \in \mathrm{Con}\,(X)$ (see [**B**] or [**D**]). Now $Uf = Vf + P_M Uf = Vf + \varphi\cdot Uf$ a.e. ξ since $t \mapsto Uf\circ X_t$ is constant a.s. P^ξ as noted earlier. Thus $Uf = Vf$ a.e. $\tilde{\xi}$ and so

$$\tilde{\xi}(0 < Vf < \infty) = \tilde{\xi}(0 < Uf < \infty) \leq \xi(0 < Uf < \infty) = 0.$$

Thus $\tilde{\xi} \in \mathrm{Con}\,(X, M)$ as desired.

(ii) $\xi \in \mathrm{Dis}\,(X)$: We first establish (4.2) in the special case $u = 1$, in which case $\tilde{1} = 1 - \varphi = \psi$ (say). Since $\xi \in \mathrm{Dis}\,(X)$, as noted earlier there is a random time $S^* \in \mathcal{G}^0$ such that $t + S^* \circ \sigma_t = S^*$ for all $t \in \mathbb{R}$, and $Q_\xi(S^* \notin \mathbb{R}) = 0$. We shall also need the sequence (S_n) of $(\mathcal{G}^0_{t+})$-stopping times constructed in [**FM**; (4.4)] so as to satisfy (a) $t + S_n \circ \sigma_t = S_n$ for all $t \in \mathbb{R}$, (b) $\alpha < S_n < \beta$ if $S_n < +\infty$, (c) $S_n \downarrow \alpha$ as $n \to \infty$ a.s. Q_ξ. As noted earlier, $N_{\beta-} = \hat{N}_{\alpha+} = \lim_{t \downarrow \alpha} M_{\zeta-} \circ \theta_t$. Thus by [**F**; (4.4)],

$$L(\xi, 1) = Q_\xi(0 < S^* < 1),$$

$$L(\xi, P_M 1) = Q_\xi(1 - N_{\beta-}; 0 < S^* < 1).$$

To prove (4.2) in the present case we must therefore show that

$$L^M(\tilde{\xi}, \tilde{1}) = Q_\xi(N_{\beta-}; 0 < S^* < 1). \tag{4.12}$$

To this end note that $N_{\beta-}$ is (σ_t)-invariant, so by [**F**; (2.4)] the R.H.S. of (4.12) may be written

$$\lim_{n \to \infty} Q_\xi(N_{\beta-}; 0 < S^* < 1,\ S_n \in \mathbb{R}) = \lim_{n \to \infty} Q_\xi(N_{\beta-};\ 0 < S_n < 1), \tag{4.13}$$

since $Q_\xi(S^* \notin \mathbb{R}) = 0$. Now $\alpha < S_n < \beta$ a.s. Q_ξ on $\{S_n < 1\}$, so $N_{\beta-} = N_{S_n} M_{\zeta-} \circ \theta_{S_n}$ a.s. Q_ξ. The S_n being stopping times, we have

$$Q_\xi(N_{\beta-};\ 0 < S_n < 1) = Q_\xi(N_{S_n} \psi \circ Y_{S_n};\ 0 < S_n < 1) \tag{4.14}$$

by the strong Markov property of (Y_t, Q_ξ). (Recall that $\psi = \tilde{1} = P^{\cdot}(M_{\zeta-})$.) Now if $t > S_n$ then $S_n \circ k_t = S_n$ and $Y_{S_n} \circ k_t = Y_{S_n}$. Also, by the construction of S_n in [**FM**], $\{S_n \circ k_t = +\infty\} = \{S_n \geq t\}$. Since $N_{S_n} = -\int_{]S_n, \infty]} dN_t$ $(N_\infty = 0)$, using (4.14) we obtain

$$Q_\xi(N_{\beta-};\ 0 < S_n < 1) = Q_\xi \left(- \int_{]\alpha, \infty]} (\psi \circ Y_{S_n} 1_{]0,1[}(S_n)) \circ k_t \, dN_t \right).$$

Theorem 4.12(iii) of [**T**] states that his last expression is

$$Q^M_{\tilde\xi}(\psi\circ Y_{S_n};\ 0 < S_n < 1), \tag{4.15}$$

where $Q^M_{\tilde\xi}$ is the Kuznetsov measure for (X, M) and $\tilde\xi \in \mathrm{Exc}(X, M)$. (This may be verified by a comparison of finite dimensional distributions.) It is easy to check that $S_n \downarrow \alpha$ a.s. $Q^M_{\tilde\xi}$, so applying [**F**; (2.4)] as in (4.13), the expression in (4.15) tends to

$$L^M(\tilde\xi, \psi) = L^M(\tilde\xi, \tilde 1)$$

as $n \to \infty$. This combined with (4.12) and (4.13) yields (4.2) for $u = 1$.

In proving (4.2) for general $u \in S(X)$ we first consider the case $u < \infty$ a.e. ξ. We shall reduce this case to that previously considered by means of the u-transform of X. We refer the reader to [**GSt**] for a discussion of the relevant properties of u-transforms, and to [**Sh**; §62] for a complete discussion. According to [**GSh**; (6.19)], given $u \in S(X)$ there is a *Borel* measurable $\overline{u} \in S(X)$ such that $u = \overline{u}$ off an M-polar set. Each of the terms in (4.2) is unchanged if u is replaced by $\overline{u}$, so in the sequel we shall assume without loss of generality that u is Borel. The u-transform of X is denoted by $X^{(u)}$ and is the Borel right process on the state space $E_u = \{0 < u < \infty\}$ with semigroup $P^{(u)}_t f = u^{-1}P_t(uf)$. In general the superscript (u) will indicate objects defined relative to $X^{(u)}$. (Two exceptions are $P^{x/u}$, the law of $X^{(u)}$ started at $x \in E_u$, and L_u, the energy functional of $X^{(u)}$.)

The following result is well-known for hitting times (and for this result u need not be Borel).

(4.16) PROPOSITION. *Fix* $u \in S(X)$.

(a) $u \cdot P^{(u)}_M 1 = P_M u$ *on* $\{u < \infty\}$;

(b) *If* $\xi \in \mathrm{Exc}\,(X)$ *and* $\xi(u = \infty) = 0$, *then* $R^{(u)}_M(u\cdot\xi) = u \cdot R_M\xi$.

Proof. In this proof only we write K^q for the operator P_M^q taken relative to $X^{(u)}$ and W^q for the resolvent of $X^{(u)}$. As is well-known

$$u(x)P^{x/u}(F;\ t<\zeta)=P^x(F\cdot u\circ X_t),\quad F\in b\mathcal{F}^u_{t+},$$

provided $u(x)<\infty$. In this case,

$$\begin{aligned}(4.17)\qquad u(x)K^qW^qf(x)&=u(x)P^{x/u}\int_0^\infty e^{-qt}f\circ X_t(1-M_t)\,dt\\ &=P_M^qU^q(uf)(x).\end{aligned}$$

If $q>0$ then there are bounded positive f_n such that $W^qf_n\uparrow 1$ on E_u. But then $U^q(f_nu)\uparrow u$ on $\{u<\infty\}$. Moreover, if $u(x)<\infty$ then $P_M^q(x,\{u=\infty\})=0$. Replacing f by f_n in (4.17) and letting $n\to\infty$ we find that $u\cdot K^q1=P_M^qu$ on $\{u<\infty\}$. Passing to the limit as $q\downarrow 0$ establishes point (a). With the help of (4.7), point (b) follows exactly as in [**GSt**; (5.4ii)]. ∎

Now fix a (Borel measurable) $u\in S(X)$ with $\xi(u=\infty)=0$. We apply (4.2) with $u=1$, X replaced by $X^{(u)}$, and ξ replaced by $u\cdot\xi\in\mathrm{Dis}\,(X^{(u)})$:

$$(4.18)\qquad L_u(u\xi,1)=(L_u)^M(\widetilde{u\xi}^{(u)},\tilde{1}^{(u)})+L_u(R_M^{(u)}(u\xi),1).$$

But $L_u(u\xi,1)=L(\xi,u)$ by [**GSt**; (4.10)]; combining this with (4.16b) shows that the third term in (4.18) reduces to $L(R_M\xi,u)$. It remains to show that the second term in (4.18) equals $L^M(\tilde\xi,\tilde u)$. Put $u^*=u|_{E_M}$. Then $u^*\in S(X,M)$ and it is easy to check that $(X^{(u)},M)=(X,M)^{(u^*)}$, hence $(L_u)^M=(L^M)_{u^*}$. By (4.16b),

$$\widetilde{u\xi}^{(u)}=u\xi-R_M^{(u)}(u\xi)=u\xi-uR_M\xi=u\tilde\xi=u^*\tilde\xi,$$

since $\tilde\xi$ is carried by E_M. Also, by (4.16a) and the fact that $P_M^{(u)}1=1$ off $E_M^{(u)}=E_M\cap E_u$,

$$u^*\tilde{1}^{(u)}=u-uP_M^{(u)}1=u-P_Mu=\tilde u$$

on $E_M \cap \{u < \infty\}$, hence a.e. $\tilde{\xi}$. Combining these observations we see that

$$(L_u)^M(\widetilde{u\xi}^{(u)}, \tilde{1}^{(u)}) = L^M(\tilde{\xi}, \tilde{u}),$$

and (4.2) follows for $\xi \in \mathrm{Dis}\,(X)$ if $u < \infty$ a.e. ξ.

Finally, consider $\xi \in \mathrm{Dis}\,(X)$, $u \in S(X)$ and suppose that $\xi(u = \infty) > 0$. First note that $L(\xi, u) = \infty$. Indeed, choosing (μ_n) such that $\mu_n U \uparrow \xi$, we have $\mu_n U(u = \infty) > 0$ for all large n. But $\{u < \infty\}$ is absorbing so $U(x, \{u = \infty\}) = 0$ if $u(x) < \infty$. It follows that $\mu_n\{u = \infty\} > 0$ and that $L(\xi, u) =\uparrow \lim \mu_n(u) = \infty$. Thus (4.2) will follow in the present case provided $\xi(P_M u = \infty) > 0$. By way of contradiction, assume that $\xi(P_M u = \infty) = 0$. Let $B = \{u = \infty\}$. Since $P_M u = u$ off E_M, $\xi(E_M \cap B) > 0$. Clearly $E_M \cap \{P_M u < \infty\} \subset \{u < \infty\}^r$ (the set of regular points for $\{u < \infty\}$), and since $\{u < \infty\}$ is absorbing, $E_M \cap \{P_M u < \infty\} \cap B^r = \emptyset$. Thus $\xi(E_M \cap B^r) = 0$. But $B^r \subset B$ and $\xi(B \backslash B^r) = 0$ ($B \backslash B^r$ is semipolar). Thus $\xi(E_M \cap B) = 0$ and we have our contradiction. The proof of (4.2) is at long last complete. $\square$

(4.19) Remarks. We conclude with a brief indication of how Theorem (4.1) may be extended to general right processes. Roughly speaking, given $\xi \in \mathrm{Exc}\,(X)$, we produce a Borel right process X^* with the same finite dimensional distributions as X for ξ a.e. starting point. As far as formula (4.2) is concerned the processes X and X^* are "equivalent". Applying Theorem (4.1) to X^* we thereby obtain formula (4.2) for X.

Passing to the details, let $(\overline{X}_t, \overline{P}_t, \overline{E})$ be a Ray compactification of X as in [**Sh**; §39]. Fix $\xi \in \mathrm{Exc}\,(X)$ and choose $E_0 \in \overline{\mathcal{E}}$ such that $E_0 \subset E$ and $\xi(E \backslash E_0) = 0$. Let $E^* = D \cap \{x \in \overline{E}\colon \overline{U}^1(x, \overline{E} \backslash E_0) = 0\}$. Then $(E^*, \overline{\mathcal{E}} \cap E^*)$ is a Lusin space, $\xi(E \backslash E^*) = 0$, and E^* is absorbing for $\overline{X}$. It follows that X^*, the restriction of $\overline{X}$ to E^*, is a Borel right process and that $E^* \backslash E$ is quasi-polar for X^* (see [**Sh**; (39.15)]). In addition, $E \backslash E^*$ is ξ-polar for X.

If $\eta \in \mathrm{Exc}\,(X)$ and $\eta(E\backslash E^*) = 0$, then η may be regarded as a measure η^* on E^*; as such $\eta^* \in \mathrm{Exc}\,(X^*)$. Similarly, if $u \in S(X)$ then $u^*(x) :=\uparrow \lim_{t\downarrow 0} \overline{P}_t u(x)$, $x \in E^*$, defines an element u^* of $S(X^*)$ such that $u^* = u$ on $E \cap E^*$. Writing L^* for the energy functional of X^*, we have $L^*(\eta^*, u^*) = L(\eta, u)$.

Now given an exact MF, M, of X, there exists an exact MF, M^*, of X^*, such that (using the obvious notation)

$$(P_M u)^* = P_{M^*}(u^*), \quad (R_M \eta)^* = R_{M^*}(\eta^*),$$

provided $u \in S(X)$, $\eta \in \mathrm{Exc}\,(X)$ with $\eta(E\backslash E^*) = 0$. We now apply Theorem (4.1) to X^* and the elements M^*, ξ^*, and u^*, and then verify that each term in (4.2) is unchanged if the $*$'s are dropped, whence (4.2) for X, M, ξ, and u. This task is routine, if lengthy, and is left to the interested reader.

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