On a Problem of Kingman

P. J. Fitzsimmons

Let $(E, \mathcal{E})$ be a Lusin metrizable space and let $\Phi$ be a random measure on $(E, \mathcal{E})$. More precisely, $\Phi$ is a random variable, defined on some probability space $(\Omega, \mathcal{F}, P)$, with values in the (positive, $\sigma$-additive) measures on $(E, \mathcal{E})$. We assume that $\Phi$ is completely random [3], in the sense that $\{\Phi(A_n) : n \in \mathbb{N}\}$ is an independent sequence of random variables whenever $\{A_n : n \in \mathbb{N}\}$ is a collection of pairwise disjoint $\mathcal{E}$-measurable sets.

It is shown in [3], under a mild $\sigma$-finiteness condition, that such a random measure $\Phi$ can be decomposed as

$$\Phi = \Phi_f + \Phi_d + \Phi_0,$$

where $\Phi_f$ is a purely atomic measure with atoms of independent sizes located at the points of a deterministic subset of $E$, and $\Phi_d$ is a non-atomic deterministic measure. The remaining component $\Phi_0$ is the most interesting of the three, and is the subject of most of the discussion in [3]. In particular, it is shown there that $\Phi_0$ is equal in distribution to a purely atomic measure $\Phi_*$. In [4] an argument due to D. Blackwell [1] is adapted to prove that $\Phi_0$ is itself purely atomic (with probability 1), under a broad additional condition. Our aim in this note is to point out that there is a simple direct argument showing that $\Phi_0$ is purely atomic in general.

For simplicity, we assume in what follows that $\mathbb{P}[\Phi(E) < \infty] = 1$. This finiteness condition is stronger than the $\sigma$-finiteness condition mentioned earlier; in particular, the decomposition (1) is valid.

Let $M$ denote the class of finite measures on $(E, \mathcal{E})$, and let $M_a$ denote the subclass of purely atomic measures. Let $\mathcal{M}$ be the $\sigma$-algebra on $M$ generated by the maps $\mu \mapsto \mu(B)$, $B \in \mathcal{E}$. In saying that $\Phi_0$ and $\Phi_*$ have the same distribution, we mean that $(\Phi_0(B_1), \Phi_0(B_2), \ldots, \Phi_0(B_n))$ has the same distribution as $(\Phi_*(B_1), \Phi_*(B_2), \ldots, \Phi_*(B_n))$ for all $n$-tuples $(B_1, B_2, \ldots, B_n)$ of elements of $\mathcal{E}$ and all $n \in \mathbb{N}$. It then follows from the monotone class theorem that

$$\mathbb{P}[\Phi_0 \in C] = \mathbb{P}[\Phi_* \in C], \quad \forall C \in \mathcal{M}.\tag{2}$$

In view of (2), we need only verify that $M_a$ is $\mathcal{M}$-measurable. But this is an immediate consequence of Lemma 2.3 on page 20 of [2]. Indeed, each $\mu \in M$ admits a unique decomposition $\mu = \mu_d + \mu_a$ into diffuse and purely atomic parts, and the mapping $\mu \mapsto \mu_d$ is $\mathcal{M}$-measurable. Thus $M_a = \{\mu \in M : \mu_d(B) = 0, \forall B \in \mathcal{E}\}$ is $\mathcal{M}$-measurable because $\mathcal{E}$ is countably generated.

References