On the Laplace Transform
of a Continuous Additive Functional

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The theorem below was established in [1] in the context of a stable Markov chain on \( \{0, 1, 2, \ldots \} \). We present here a simple proof based on the domination principle.

Let \( X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x) \) be a right Markov process with state space \( E \), finite lifetime \( \zeta \), and cemetery state \( \Delta \). Let \( A = (A_t)_{t \geq 0} \) be a continuous additive functional of \( X \) such that \( A_\zeta < \infty \) almost surely. Define \( \varphi(x) := \mathbb{P}^x[\exp(-A_\zeta)] \), with the understanding that \( \varphi(\Delta) = 1 \). Observe that

\[
1 - \exp(-A_\zeta) = \int_0^\zeta \exp(-A_\zeta \circ \theta_t) \, dA_t,
\]

whence the identity

\[
1 - \varphi(x) = \mathbb{P}^x \int_0^\zeta \varphi(X_t) \, dA_t = U_A \varphi(x), \quad x \in E_\Delta,
\]

where \( U_A \) is the potential operator associated with \( A \): \( U_A \varphi(x) := \mathbb{P}^x \int_0^\zeta \varphi(X_t) \, dA_t \). In particular,

\[
M_t^x := \mathbb{E}^x \left[ \int_0^\zeta \varphi(X_s) \, dA_s \bigg| \mathcal{F}_t \right] = \int_0^t \varphi(X_s) \, dA_s + 1 - \varphi(X_t), \quad t \geq 0,
\]

is a uniformly integrable martingale under \( \mathbb{P}^x \), for each \( x \in E \). The intuitive content of this statement becomes clearer when \( A \) has the simple form \( A_t = \int_0^t a(X_s) \, ds \) for some bounded positive Borel function \( a \). Writing \( L \) for the infinitesimal generator of \( X \), we see that the martingale property of \( M_t^x \) amounts to saying that \( L \varphi = a \cdot \varphi \). The theorem stated below asserts that \( \varphi \) is the maximal solution of this equation with values in \([0, 1]\).

**Theorem.** Let \( w : E_\Delta \to [0, 1] \) be a finely continuous function, and suppose that

\[
M_t := w(X_t) - \int_0^t w(X_s) \, dA_s
\]

is a uniformly integrable martingale under \( \mathbb{P}^x \), for each \( x \in E \). Then \( w \leq \varphi \).

**Proof.** Define \( h := w + U_A w \). It is easy to check that \( h \) is harmonic, in the sense that \( \mathbb{P}^x[h(X_T)] = h(x) \) for each stopping time \( T \). Observe that

\[
U_A(\varphi - w) = 1 - h - (\varphi - w) \leq 1 - h \quad \text{on} \ \{\varphi > w\}.
\]

The harmonicity of \( 1 - h \) and the domination principle now imply that

\[
U_A(\varphi - w) \leq 1 - h \quad \text{on} \ E.
\]

That is,

\[
-(\varphi - w) \leq 0 \quad \text{on} \ E,
\]

which means that \( w \leq \varphi \) on all of \( E_\Delta \). \( \Box \)

**Reference**