

Supplement on the Monotone Class Theorem

In dealing with integrals, the following “functional” form of the Monotone Class Theorem is often useful.

(1) Theorem. Let \mathcal{K} be a collection of bounded real-valued functions on Ω that is closed under the formation of products (i.e., if $f, g \in \mathcal{K}$ then $fg \in \mathcal{K}$), and let \mathcal{B} be the σ -algebra generated by \mathcal{K} . Let $\mathcal{H} \supset \mathcal{K}$ be a vector space (over \mathbf{R}) of bounded real-valued functions on Ω such that (a) \mathcal{H} contains the constant functions and (b) if $(f_n) \subset \mathcal{H}$ with $\sup_n \sup_\omega |f_n(\omega)| < +\infty$ and if $0 \leq f_1 \leq f_2 \leq \dots \leq f_n \leq \dots$, then $f := \lim_n f_n \in \mathcal{H}$. Under these conditions, \mathcal{H} contains every bounded \mathcal{B} -measurable real-valued function on Ω .

The proof of this theorem relies on the following

(2) Lemma. If \mathcal{H} is as in the statement of Theorem (1), then \mathcal{H} is closed under uniform convergence.

Proof. Suppose $(f_n) \subset \mathcal{H}$ and $f_n \rightarrow f$ uniformly on Ω . (That is, $\lim_n \sup_\omega |f_n(\omega) - f(\omega)| = 0$.) By passing to a subsequence if necessary, we can arrange that $\|f_{n+1} - f_n\|_\infty \leq 2^{-n}$. Define $g_n := f_n - 2^{1-n} + 2$. Then $g_n \in \mathcal{H}$ since \mathcal{H} is a vector space containing the constant functions, and

$$\sup_\omega |g_n(\omega)| \leq \sup_\omega |f_n(\omega)| + 2,$$

so the sequence (g_n) is uniformly bounded. Also, $g_{n+1} - g_n = f_{n+1} - f_n + 2^{-n} \geq 0$ ($n = 1, 2, \dots$) and $g_1 = f_1 + 2^{-1} \geq 0$. It follows that (g_n) is a (uniformly bounded) increasing sequence, with $\lim_n g_n = \lim_n f_n + 2 = f + 2$. But $\lim_n g_n \in \mathcal{H}$, hence so is $f = \lim_n g_n - 2$. \square

Proof of (1). Owing to the closure properties of \mathcal{H} and the fact that every \mathcal{B} -measurable function is the pointwise limit of an increasing sequence of simple functions, it suffices to show that \mathcal{H} contains the indicator 1_D of every $D \in \mathcal{B}$. Define $\mathcal{L} := \{D \in \mathcal{B} : 1_D \in \mathcal{H}\}$. It is easy to see that \mathcal{L} is a λ -system. We are going to show that \mathcal{L} contains a π -system \mathcal{P} generating \mathcal{B} . In view of the Monotone Class Theorem, this will imply $\mathcal{L} \supset \mathcal{B}$, whence $\mathcal{L} = \mathcal{B}$.

Let \mathcal{A}_0 denote the algebra* generated by \mathcal{K} ; since \mathcal{K} is already closed under products, \mathcal{A}_0 is simply the linear span of \mathcal{K} . Consequently, $\mathcal{A}_0 \subset \mathcal{H}$. By Lemma (2), the uniform closure \mathcal{A} of \mathcal{A}_0 is also contained in \mathcal{H} . Referring to the standard proof of Weierstrass' Theorem, we see that if $f \in \mathcal{A}$, then $|f| \in \mathcal{A}$ as well. Consequently, if $f, g \in \mathcal{A}$, then $f \vee g = [|f - g| + f + g]/2$ and $f \wedge g = [f + g - |f - g|]/2$ are elements of \mathcal{A} . Now fix $f \in \mathcal{A}$ and $b \in \mathbf{R}$. Then for each $n \in \mathbf{N}$ the function $\varphi_n := [n(f - b)^+] \wedge 1$ is an element of \mathcal{A} , hence an element of \mathcal{H} . As $n \rightarrow \infty$, φ_n increases pointwise to $1_{\{f > b\}}$. Thus, since \mathcal{H} is closed under bounded monotone convergence, $1_{\{f > b\}} \in \mathcal{H}$; this means that $\{f > b\} \in \mathcal{L}$. More generally, if $\{f_1, f_2, \dots, f_m\}$ is a finite sequence of functions from \mathcal{A} and if $\{b_1, b_2, \dots, b_m\}$ is a finite sequence of real numbers, then the function

$$g_n := \prod_{k=1}^m [n(f_k - b_k)^+] \wedge 1$$

* An algebra (of functions) is a vector space that is also closed under the formation of products.

is an element of \mathcal{A} , and the sequence $\{g_n\}$ increases boundedly and pointwise to $\prod_{k=1}^m 1_{\{f_k > b_k\}}$, which is therefore an element of \mathcal{H} , as before. This function is the indicator of the set $B := \bigcap_{k=1}^m \{\omega : f_k(\omega) > b_k\}$, so $B \in \mathcal{L}$. It follows that \mathcal{L} contains the π -system \mathcal{P} consisting of finite intersections of sets of the form $f^{-1}(I)$, where $f \in \mathcal{A}$ and $I \subset \mathbf{R}$ is an open right-halfline. Since $\mathcal{K} \subset \mathcal{A}$, the σ -algebra generated by \mathcal{P} is \mathcal{B} ($= \sigma(\mathcal{K})$). We have constructed a π -system generating \mathcal{B} and contained in \mathcal{L} , as desired. \square

(3) Exercise. Let (Ω, \mathcal{B}) be a measurable space. Let \mathcal{K} be a collection of bounded \mathcal{B} -measurable real-valued functions on Ω such that \mathcal{B} is the σ -algebra generated by \mathcal{K} . Assume that \mathcal{K} is closed under the formation of products. Let P and Q be two probability measures on (Ω, \mathcal{B}) such that $\int X dP = \int X dQ$ for all $X \in \mathcal{K}$. Prove that $\int X dP = \int X dQ$ for every bounded \mathcal{B} -measurable random variable X . In particular, $P = Q$ on \mathcal{B} . [Hint: Take \mathcal{H} to be the class of bounded \mathcal{B} -measurable random variables X such that $\int X dP = \int X dQ$, and apply Theorem (1).]

(4) Example. Let (X, \mathcal{M}) and (Y, \mathcal{N}) be measurable spaces, and recall that $\mathcal{M} \otimes \mathcal{N}$ denotes the product σ -algebra on the cartesian product $X \times Y$. More precisely, $\mathcal{M} \otimes \mathcal{N}$ is the σ -algebra on $X \times Y$ generated by the projections π_1, π_2 , where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for $(x, y) \in X \times Y$. Let \mathcal{K} denote the set of functions of the form $(x, y) \mapsto f(x)g(y)$, where f (resp. g) is a bounded real-valued \mathcal{M} -measurable (resp. \mathcal{N} -measurable) function on X (resp. Y). Clearly the σ -algebra generated by \mathcal{K} is just $\mathcal{M} \otimes \mathcal{N}$. Now let \mathcal{H} be the set of bounded real-valued $\mathcal{M} \otimes \mathcal{N}$ -measurable functions h such that $x \mapsto h(x, y)$ is \mathcal{M} -measurable for each fixed $y \in Y$. It is a simple matter to check that \mathcal{H} is a vector space satisfying the conditions of Theorem (1). As a consequence of that result, \mathcal{H} is precisely the class of *all* bounded real-valued $\mathcal{M} \otimes \mathcal{N}$ -measurable functions. This is an alternative proof of one of the measurability assertions in Tonelli's theorem, at least for bounded functions. A truncation argument reduces the general case to the bounded case.

(5) Exercise. Let (Ω, \mathcal{B}, P) be a probability space, and let X and Y be random variables defined on (Ω, \mathcal{B}) . Suppose that

$$(6) \quad E[f(X)g(Y)] = E[f(X)g(X)]$$

for every pair (f, g) of bounded continuous functions from \mathbf{R} to \mathbf{R} . Prove that $P[X = Y] = 1$. [Hint: Use Theorem (1) to show that $E[h(X, Y)] = E[h(X, X)]$ for every bounded $\mathcal{B}(\mathbf{R}^2)$ -measurable real-valued function h . Then consider $h = 1_\Delta$, where $\Delta = \{(x, x) : x \in \mathbf{R}\}$ is the "diagonal" in \mathbf{R}^2 .]

The following variant of Theorem (1) is sometimes useful. The proof is quite similar to that of Theorem (1), and so is omitted.

(7) Theorem. *Let \mathcal{C} be an algebra of bounded real-valued functions on Ω that contains the constant functions, and let \mathcal{B} be the σ -algebra generated by \mathcal{C} . Let $\mathcal{H} \supset \mathcal{C}$ be a set of bounded real-valued functions on X that is closed under bounded monotone convergence and uniform convergence. Under these conditions, \mathcal{H} contains every bounded \mathcal{B} -measurable real-valued function on Ω .*

As an application of Theorem (7), you are asked to prove a functional form of Exercise 5 from Chapter 2.

(8) Exercise. Let (S, d) be a metric space, let $BC(S)$ denote the class of bounded continuous real-valued functions on S , and let $\mathcal{B}(S)$ denote the Borel σ -field on S ; thus $\mathcal{B}(S)$ is the σ -field generated by the open subsets of S .

- (a) Verify that $\mathcal{B}(S)$ is the σ -algebra generated by $BC(S)$.
- (b) Let μ be a probability measure on $(S, \mathcal{B}(S))$ and let f be a bounded real-valued $\mathcal{B}(S)$ -measurable function. Using Theorem (7), show that for each $\epsilon > 0$ there exists $g \in BC(S)$ such that $\int |f - g| d\mu \leq \epsilon$. [Hint: Take \mathcal{H} to be the class of bounded real-valued $\mathcal{B}(S)$ -measurable functions for which the asserted approximation property holds, and take $\mathcal{C} = BC(S)$. Show that \mathcal{H} has the required closure properties for Theorem (7) to apply.]

(9) Example. Let $\{X_t : t \in \mathbf{T}\}$ be a collection of random variables, indexed by \mathbf{T} , defined on a common measurable space (Ω, \mathcal{B}) . Let $\mathcal{X} \subset \mathcal{B}$ denote the σ -field generated by $\{X_t : t \in \mathbf{T}\}$.

Claim: If $F : \Omega \rightarrow \mathbf{R}$ is \mathcal{X} -measurable, then there is a sequence $\{t_1, t_2, \dots\} \subset \mathbf{T}$ and a $\mathcal{B}(\mathbf{R}^{\mathbf{N}})$ -measurable function $f : \mathbf{R}^{\mathbf{N}} \rightarrow \mathbf{R}$ such that

$$(10) \quad F(\omega) = f(X_{t_1}(\omega), X_{t_2}(\omega), \dots), \quad \forall \omega \in \Omega.$$

To see the Claim apply Theorem (1) with \mathcal{K} equal to the class of functions of the form

$$\omega \mapsto \prod_{k=1}^n f_k(X_{t_k}(\omega)),$$

where $n \in \mathbf{N}$, $t_k \in \mathbf{T}$, and each $f_k : \mathbf{R} \rightarrow \mathbf{R}$ is bounded and $\mathcal{B}(\mathbf{R})$ -measurable. Observe that $\sigma(\mathcal{K}) = \mathcal{X}$. Take \mathcal{H} to be the class of bounded \mathcal{X} -measurable functions for which a representation like (10) holds. It should be clear that \mathcal{H} is a vector space containing the constant functions. Let $\{F_n\}_{n \in \mathbf{N}}$ be a uniformly bounded increasing sequence of elements of \mathcal{H} . Since a countable union of countable sets is itself countable, we may suppose that there is a single sequence $\{t_1, t_2, \dots\}$ such that

$$F_n(\omega) = f_n(X_{t_1}(\omega), X_{t_2}(\omega), \dots), \quad \forall n \in \mathbf{N}, \omega \in \Omega,$$

where each f_n is bounded and $\mathcal{B}(\mathbf{R}^{\mathbf{N}})$ -measurable. Since $\{F_n\}$ is uniformly bounded, we can suppose that there is a constant M such that $|f_n(\mathbf{x})| \leq M$ for all $n \in \mathbf{N}$ and all $\mathbf{x} \in \mathbf{R}^{\mathbf{N}}$. Define $f : \mathbf{R}^{\mathbf{N}} \rightarrow \mathbf{R}$ by

$$f(\mathbf{x}) := \liminf_{n \rightarrow \infty} f_n(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^{\mathbf{N}}.$$

Then f is bounded in magnitude by M and is $\mathcal{B}(\mathbf{R}^{\mathbf{N}})$ -measurable. Moreover, because $F_n(\omega)$ increases with n ,

$$F(\omega) := \lim_n F_n(\omega) = \lim_n \inf_n F_n(\omega) = \lim_n \inf_n f_n(X_{t_1}(\omega), X_{t_2}(\omega), \dots) = f(X_{t_1}(\omega), X_{t_2}(\omega), \dots),$$

so F admits a representation as in (10). That is, \mathcal{H} is closed under bounded monotone convergence. By Theorem (1), \mathcal{H} contains every bounded \mathcal{X} -measurable real-valued function on Ω . This proves the Claim.