Math 286, Fall 2004

Martingale Representation Theorem

The following fundamental result is due to H. Kunita & S. Watanabe. Throughout, $(B_t)_{t\ge 0}$ is a standard Brownian motion (with $B_0 = 0$) defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $(\mathcal{F}_t)_{t\ge 0}$ is the filtration generated by (B_t).

Theorem. *If $F \in L^2(\mathcal{F}_1)$, then there is a unique element ϕ of $\mathcal{V}$ such that $\phi_s \equiv 0$ for $s > 1$ and*

$$F = \mathbf{E}[F] + \int_0^1 \phi_s \, dB_s \tag{1}$$

almost surely.

Proof. Substituting $F - \mathbf{E}[F]$ for F, we reduce to the case $\mathbf{E}[F] = 0$, which is henceforth assumed.

Let I denote the vector space of random variables of the form $\int_0^1 \phi_s \, dB_s$, where $\phi \in \mathcal{V}$ vanishes on $\Omega \times (1,\infty)$, and define $L := \{G \in L^2(\mathcal{F}_1) : \mathbf{E}[G] = 0\}$. Evidently I is a closed subspace of L, and our goal is to show that $I = L$. To this end it suffices to show that the only element of L that is orthogonal to every element of I is 0.

So fix $F \in L$ and suppose that $\mathbf{E}[FJ] = 0$ for every stochastic integral $J = \int_0^1 \phi_s \, dB_s$ in I. Given times $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n \le 1$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, define an (elementary) element of $\mathcal{V}$ by

$$K_s := i \sum_{j=1}^n \lambda_j 1_{(t_{j-1}, t_j]}(s), \qquad 0 \le s \le 1,$$

where $i = \sqrt{-1}$. Let M be the martingale $t \mapsto \int_0^t K_s \, dB_s$, and define

$$X_t := \exp\left(M_t - \tfrac{1}{2}\langle M\rangle_t\right),$$

where $\langle M\rangle_t = \int_0^t K_s^2 \, ds$ is the "quadratic variation" process of M. As discussed in class, Itô's formula implies that X is a martingale with $\sup_{0\le t\le 1} \mathbf{E}[X_t^2] < \infty$; moreover,

$$X_t = 1 + \int_0^t X_s K_s \, dB_s, \qquad \forall t \in [0,1]$$

almost surely. Since $|K_s(\omega)| \le \sum_{j=1}^n |\lambda_j|$, it follows that $XK \in \mathcal{V}$ and so the random variable $X_1 - 1$ is an element of I. Thus, $\mathbf{E}[F(X_1 - 1)] = 0$, so

$$0 = \mathbf{E}[F] = \mathbf{E}[F X_1]. \tag{2}$$

Using the fact that $\langle M\rangle_1$ is non-random, we deduce from (2) that

$$\mathbf{E}[F \exp(M_1)] = 0;$$

more explicitly,

$$\mathbf{E}\left[F \prod_{j=1}^n e^{i\lambda_j (B_{t_j} - B_{t_{j-1}})}\right] = \mathbf{E}\left[F \exp\left(\sum_{j=1}^n i\lambda_j (B_{t_j} - B_{t_{j-1}}\right)\right] = 0. \tag{3}$$

Invoking the uniqueness theorem for characteristic functions (or Weierstrass' theorem) we deduce from (3) that

$$\mathbf{E}\left[F \prod_{j=1}^n f_j(B_{t_j} - B_{t_{j-1}})\right] = 0 \tag{4}$$

for all bounded continuous functions $f_1, f_2, \ldots, f_n$. The monotone class theorem now allows us to deduce from (4) that

$$\mathbf{E}[FG] = 0$$

for every bounded $\mathcal{F}_1$-measurable function G. In particular,

$$\mathbf{E}[F \arctan(F)] = 0$$

and so $F = 0$ almost surely because $x \arctan(x) > 0$ unless $x = 0$. ∎

Straightforward localization arguments lead to the following

Corollary. *If M is a local martingale with right-continuous paths, then there is an essentially unique process $\phi \in \mathcal{V}_{\text{loc}}$ such that*

$$M_t = \int_0^t \phi_s \, dB_s, \qquad \forall t \geq 0, \tag{5}$$

almost surely. In particular, every right-continuous local martingale of the Brownian motion has continuous paths.

Proof. We only comment on some aspects of the proof. The uniqueness of ϕ is in the sense of $\mathcal{V}_{\text{loc}}$. Concerning the final assertion, suppose M is a right-continuous local martingale. Then by (5), M agrees almost surely with a stochastic integral, which has continuous sample paths. It follows that M has continuous sample paths. ∎

One drawback of the theorem is that is provides no clue as to how one might find the process ϕ appearing in the representation formula (1). Before proceeding with some examples, a general remark is in order. Let F be as in (1), and assume for simplicity that $\mathbf{E}[F] = 0$. Since the Brownian filtration $(\mathcal{F}_t)$ is right continuous, the martingale $\mathbf{E}[F|\mathcal{F}_t]$ admits a right-continuous version F_t. Taking conditional expectations in (1) we find that

$$F_t = \int_0^t \phi_s \, dB_s, \qquad \forall t \in [0,1],$$

almost surely. Now choose $K \in \mathcal{V}$, and let M^K denote the martingale $\int_0^t K_s \, dB_s$. Then $F_t M_t^K - \int_0^t \phi_s K_s \, ds$ is a martingale, and so

$$\mathbf{E}[F_t M_t^K] = \int_0^t \mathbf{E}[\phi_s K_s] \, ds, \qquad 0 \leq t \leq 1. \tag{6}$$

Now in (6) take $t = 1$, and take K to be of the form $K_s = 1_A 1_{(v,1]}$, where $A \in \mathcal{F}_u$ and $0 < u < v < 1$. In this case $M_1^K = 1_A(B_1 - B_v)$, so (6) reads

$$\mathbf{E}[F(B_1 - B_v); A] = \int_v^1 \mathbf{E}[\phi_s; A] \, ds;$$

equivalently,

$$\mathbf{E}[F(B_1 - B_v)|\mathcal{F}_u] = \int_v^1 \mathbf{E}[\phi_s|\mathcal{F}_u] \, ds. \tag{7}$$

Proceeding heuristically, let us differentiate (7) with respect to v and then evaluate at $v = u$:

$$-\frac{d}{dv}\mathbf{E}[FB_v|\mathcal{F}_u]\Big|_{v=u} = \frac{d}{dv}\mathbf{E}[F(B_1 - B_v)|\mathcal{F}_u]\Big|_{v=u} = -\mathbf{E}[\phi_u|\mathcal{F}_u] = -\phi_u. \tag{8}$$

In principle, this gives us a way to compute the process ϕ, provided we can evaluate the mess on the left side of (8). This can be done for sufficiently smooth F using an "integration by parts" formula with respect to the measure $\mathbf{P}$ on the sample space of the Brownian motion. (This integration by parts transfers the derivative from B to F). The formula is beyond the scope of the present note, but a nice discussion can be found in volume 2 of "Diffusions, Markov Processes, and Martingales" by L.C.G. Rogers & D. Williams. The resulting expression for ϕ (in terms of a conditional expectation of the derivative of F) is known as Clark's Formula.

Turning to examples, consider the case $F = f(B_1)$. Assume for the moment that f is smooth and bounded. Then $F_t = P_{1-t}f(B_t)$, where

$$P_s f(x) := \mathbf{P}_x[f(B_s)] = \int_{\mathbf{R}} p_s(x,y)f(y)\,dy = \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} f(y)\,dy$$

is the transition operator for Brownian motion:

$$\mathbf{E}[f(B_{t+s})|\mathcal{F}_t] = \mathbf{E}^{B_t}[f(B_s)] = P_s f(B_t).$$

An appeal to the dominated convergence theorem shows that $P_{1-t}f(x)$ is a smooth function of $(x,t) \in \mathbf{R} \times [0,1]$; Itô's formula therefore yields

$$F_t = P_{1-t}f(B_t) = P_1 f(0) + \int_0^t P_{1-s}(f')(B_s)\,dB_s \tag{9}$$

because $[\partial/\partial t]P_{1-t}f(x) = -1/2[\partial^2/\partial x^2]P_{1-t}f(x)$. Thus, $\phi_s = P_{1-s}(f')(B_s)$ in this case. An approximation argument shows that (9) persists for general measurable f subject only to the condition that $f(B_1)$ be square integrable. An interesting consequence of (9) is the "Poincaré inequality"

$$\int_{\mathbf{R}} [f(x) - \overline{f}]^2\,\mu(dx) \le \int_{\mathbf{R}} [f'(x)]^2\,\mu(dx), \tag{10}$$

where μ denotes the standard normal distribution (which is the distribution of B_1 under $\mathbf{P}_0$) and $\overline{f} := \int_{\mathbf{R}} f(x)\,\mu(dx) = P_1 f(0)$. To see (10) use (9) to calculate the variance of F:

$$\int_{\mathbf{R}} [f(x) - \overline{f}]^2\,\mu(dx) = \mathbf{E}[[F_1 - \mathbf{E}(F_1)]^2] = \int_0^1 \mathbf{E}[[P_{1-s}(f')(B_s)]^2]\,ds. \tag{11}$$

But by the Schwarz inequality, $[P_{1-s}(f')(x)]^2 \le P_{1-s}([f']^2)(x)$, so the right side of (11) is dominated by

$$\int_0^1 \mathbf{E}[P_{1-s}([f']^2)(B_s)]\,ds = \int_0^1 P_s(P_{1-s}([f']^2))(0)\,ds = P_1([f']^2))(0) = \int_{\mathbf{R}} [f']^2\,d\mu.$$

Continuing in the same vein, let f and g be smooth bounded *increasing* functions. Develop $f(B_1)$ and $g(B_1)$ as in (9) and then take expectations:

$$\mathbf{E}[f(B_1)g(B_1)] = \mathbf{E}[f(B_1)]\,\mathbf{E}[g(B_1)] + \int_0^1 \mathbf{E}[P_{1-s}(f')(B_s)\,P_{1-s}(g')(B_s)]\,ds \ge \mathbf{E}[f(B_1))\,\mathbf{E}(g(B_1)],$$

because f' and g' are positive by hypothesis. Thus, *increasing functions of B_1 are positively correlated.*

One final example shows how (8) can be used, at least in simple cases. Take $F = \int_0^1 B_s\,ds$. As an exercise in conditional expectations with respect to Gaussian distributions, show that

$$\mathbf{E}[F(B_1 - B_v)] = \int_v^1 (s-v)\,ds = (1-v)^2/2, \qquad 0 < v < 1.$$

Consequently the left side of (8) is $-(1-u)$, and so $\phi_u = (1-u)$ in this case. That is,

$$\int_0^1 B_s\,ds = \int_0^1 (1-s)\,dB_s \tag{12}$$

since $\mathbf{E}[F] = 0$ by symmetry. Formula (12) can also be obtained directly from Itô formula.