

## Math 286, Fall 2006

### An Integral

The following integral arises in the calculation of the Laplace transform of the density function of the first passage time  $T_b$ .

**Proposition.** For  $\alpha > 0$  and  $\beta \geq 0$ ,

$$(1) \quad \int_0^\infty e^{-\alpha t - \beta/t} t^{-1/2} dt = \sqrt{\frac{\pi}{\alpha}} e^{-2\sqrt{\alpha\beta}}.$$

*Proof.* Fix  $\alpha > 0$  and view the left side of (1) as a function of  $\beta$ ; call this function  $\varphi$ . Observe that

$$(2) \quad \varphi(0) = \int_0^\infty e^{-\alpha t} t^{-1/2} dt = \alpha^{-1/2} \int_0^\infty e^{-u} u^{-1/2} du = \alpha^{-1/2} \Gamma(1/2) = \sqrt{\pi/\alpha},$$

where the second equality results from the change of variables  $u := \alpha t$ . Also, differentiating under the integral sign (justification left to the reader),

$$(3) \quad \varphi'(\beta) = - \int_0^\infty e^{-\alpha t - \beta/t} t^{-3/2} dt, \quad \beta > 0.$$

Let us make the change of variable  $s := \beta/(\alpha t)$  on the right side of (3). Then  $dt = -(\beta/\alpha) ds/s^2$ , so

$$(4) \quad \varphi'(\beta) = - \sqrt{\frac{\alpha}{\beta}} \int_0^\infty e^{-\beta/s - \alpha s} s^{-1/2} ds = - \sqrt{\frac{\alpha}{\beta}} \varphi(\beta).$$

The equality between the extreme terms in (4) is a simple first order linear differential equation; separating variables and integrating we find that

$$\varphi(\beta) = C \cdot e^{-2\sqrt{\alpha\beta}}.$$

The constant  $C$  is the initial value  $\varphi(0) = \sqrt{\pi/\alpha}$  exhibited in (2).  $\square$

**Corollary 1.** For  $\alpha > 0$  and  $\beta \geq 0$ ,

$$(5) \quad \int_0^\infty e^{-\alpha t - \beta/t} t^{-3/2} dt = \sqrt{\pi/\beta} e^{-2\sqrt{\alpha\beta}}.$$

*Proof.* Differentiate with respect to  $\beta$  in (1).  $\square$

**Corollary 2.** For  $\alpha > 0$  and  $b > 0$ ,

$$\int_0^\infty e^{-\alpha t} \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t} dt = \exp^{-b\sqrt{2\alpha}}.$$

*Proof.* This follows immediately from (5) with  $\beta = b^2/2$ .