This exam is closed book and closed notes.
You may use a calculator and your formula sheet.
Please write your section number on the cover of your bluebook.
Show your work and justify your answers.
Each problem is worth 10 points.

1. *Let X and Y be independent random variables with $\mathbf{E}(X) = 1$, $\mathbf{E}(Y) = 2$, $\mathbf{E}(X^2) = 3$, and $\mathbf{E}(Y^2) = 5$.*

(a) Compute $\mathrm{Var}(X)$ *and* $\mathrm{Var}(Y)$.

(b) Compute $\mathrm{Var}(2X + 3Y)$.

Solution.

(a) $\mathrm{Var}(X) = \mathbf{E}(X^2) - [\mathbf{E}(X)]^2 = 3 - 1^2 = 2$. Likewise, $\mathrm{Var}(Y) = 5 - 2^2 = 1$.

(b) Because X and Y are independent, $\mathrm{Var}(2X + 3Y) = \mathrm{Var}(2X) + \mathrm{Var}(3Y) = 4\mathrm{Var}(X) + 9\mathrm{Var}(Y) = 4 \cdot 2 + 9 \cdot 1 = 17$.

2. *A pair (X, Y) of discrete random variables has joint distribution $\mathbf{P}(X = x, Y = y)$ given by the following table:*

$Y \setminus X$	1	2	3	4
1	4/16	1/16	1/16	1/16
2	0	3/16	1/16	1/16
3	0	0	2/16	1/16
4	0	0	0	1/16

(a) Find the marginal probabilities $\mathbf{P}(X = x)$ and $\mathbf{P}(Y = y)$.

(b) Explain why X and Y are **not** *independent.*

Solution. (a) The marginal distribution of X is gotten by computing the column sums of the above table. These are summarized in the following table:

x	1	2	3	4
$\mathbf{P}(X = x)$	4/16	4/16	4/16	4/16

Similarly, the marginal distribution of Y is gotten by computing the row sums of the above table. These are summarized in the following table:

y	1	2	3	4
$\mathbf{P}(Y = y)$	7/16	5/16	3/16	1/16

(b) Consulting the above tables, $\mathbf{P}(X = 1, Y = 2) = 0$ but $\mathbf{P}(X = 1)\mathbf{P}(Y = 2) = (4/16) \cdot (5/16) = 5/64 \neq 0$, so X and Y are not independent.

3. *A fair coin is tossed 100 times and X is the number of heads that result. Use Chebychev's inequality to show that $\mathbf{P}(|X - 50| \leq 10) \geq .75$.*

Solution. Since X has the binomial distribution with $n = 100$ and $p = 1/2$, we have $\mu = \mathbf{E}(X) = 50$ and $\mathrm{Var}(X) = 25$. Therefore the standard deviation of X (call it σ) is equal to 5. Chebychev's inequality states that

$$\mathbf{P}[\,|X-\mu| > k\sigma] \le \frac{1}{k^2}.$$

Using this with with $k = 2$, we obtain

$$\mathbf{P}[\,|X-50| > 10] \le .25.$$

Passing to the complementary event, we find that

$$\mathbf{P}[\,|X-50| \le 10] \ge .75.$$

4. *An bowl contains 3 red balls and 5 green balls. You draw a ball at random from the bowl, note its color, and return the ball to the bowl. You repeat this sampling-with-replacement process until a red ball is drawn. Let N be the number of draws made.*

(a) Find $\mathbf{P}(N = k)$ for $k = 1, 2, 3, \ldots$.

(b) Compute $\mathbf{P}(N$ is even).

Solution. (a) $\mathbf{P}(N = k) = (5/8)^{k-1}(3/8)$ for $k = 1, 2, 3, \ldots$, because $N = k$ if and only if the first $k - 1$ draws result in green, and the k^{th} draw is red.

(b) Let's add the appropriate geometric series:

$$\mathbf{P}(N \text{ is even}) = \sum_{n=1}^{\infty} \mathbf{P}(N = 2n) = \sum_{n=1}^{\infty} (5/8)^{2n-1}(3/8) = \frac{(5/8)(3/8)}{1-(5/8)^2)} = \frac{15}{64-25} = \frac{5}{13} = .3846.$$

5. *Emma Zahn runs a bookstore, but business is not so good. The number of books that Emma sells on a given day is a random variable having the Poisson distribution with parameter $\lambda = 5$, and sales on different days are mutually independent. Find an approximate value for the probability that Emma sells at least 1760 books in a one year period. Note: Emma's bookstore is open 365 days a year. [Hint: Use the normal approximation.]*

Solution. The total number of books that Emma sells in a year, call it X, is the sum of 365 independent identically distributed random variables, each with mean and variance equal to 5. This means that $\mathbf{E}(X) = \mathrm{Var}(X) = 365 \cdot 5 = 1825$; in particular, X has standard deviation $\sigma = 42.72$. Using the normal approximation (Central Limit Theorem!):

$$\mathbf{P}[X \ge 1760] \approx 1 - \Phi\left(\frac{1760 - .5 - 1825}{42.72}\right) = 1 - \Phi(-1.53) = \Phi(1.53) = .9370.$$

[I have used the continuity correction above, since X has integer values. Without the continuity correction, one obtains $\Phi(1.52) = .9357$ for the requested probability.]

6. *X is a random variable with the continuous-type density function*

$$f(x) = \begin{cases} 4x^3, & 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $\mathbf{E}(X)$ *and* $\text{Var}(X)$.

(b) Compute $\mathbf{P}(X \le .5)$.

Solution. (a)

$$\mathbf{E}(X) = \int_0^1 x \cdot 4x^3\,dx = \int_0^1 4x^4\,dx = \frac{4x^5}{5}\Big|_0^1 = 4/5$$

and

$$\mathbf{E}(X^2) = \int_0^1 x^2 \cdot 4x^3\,dx = \int_0^1 4x^5\,dx = \frac{4x^6}{6}\Big|_0^1 = 2/3.$$

Consequently, $\text{Var}(X) = (2/3) - (4/5)^2 = 2/75$.

(b) $\mathbf{P}(X \le .5) = \int_0^{.5} 4x^3\,dx = x^4\Big|_0^{.5} = (1/2)^4 = 1/16.$