Math 280C, Spring 2005

Exponential Martingales

In what follows, $(\Omega, \mathcal{F}, \mathbf{P})$ is the canonical sample space of the Brownian motion $(B_t)_{t\ge 0}$ with $B_0 = 0$; other notation is that used in class.

Given $H \in \mathcal{L}^2_{\text{loc}}$ let M denote the associated local martingale:

$$M_t := \int_0^t H_s \, dB_s, \qquad t \ge 0. \tag{1}$$

Now define a strictly positive continuous adapted process Z by

$$Z_t := \exp\left(M_t - \tfrac{1}{2}\langle M\rangle_t\right), \qquad t \ge 0. \tag{2}$$

Clearly $Z_0 = 1$, and it follows easily from Itô's formula that

$$Z_t = 1 + \int_0^t Z_s \, dM_s = 1 + \int_0^t Z_s H_s \, dB_s. \tag{3}$$

In other words, Z solves the "stochastic differential equation" (SDE)

$$dZ_t = Z_t \, dM_t, \qquad t \ge 0, \tag{4}$$

with initial condition

$$Z_0 = 1. \tag{5}$$

For this reason we refer to Z as the *stochastic exponential* of M.

In view of (3), Z is a local martingale. Let (T_n) reduce Z. Then for each n

$$1 = \mathbf{E}[Z_0] = \mathbf{E}[Z_{t\wedge T_n}], \tag{6}$$

Because Z is a positive local martingale we can appeal to Fatou's lemma to deduce that

$$\mathbf{E}[Z_t] = \mathbf{E}[\lim_n Z_{t\wedge T_n}] \le \liminf_n \mathbf{E}[Z_{t\wedge T_n}] = 1. \tag{7}$$

Thus Z_t is integrable for each $t \ge 0$. A second application of Fatou's lemma shows that Z is a supermartingale. In particular, Z is a martingale if and only if $\mathbf{E}[Z_t] = 1$ for all $t > 0$.

Theorem 1. *Z is the unique solution of the initial value problem (4), (5).*

Proof. Let Y be a second (continuous) local martingale such that $Y_t = 1 + \int_0^t Y_s\, dM_s$ for all $t \ge 0$. Because $Z_t > 0$ for all t we can apply Itô's formula to the ratio Y/Z:

$$\begin{aligned} d(Y_t Z_t^{-1}) &= Y_t\, d(Z_t^{-1}) + Z_t^{-1}\, dY_t + d\langle Y, Z^{-1}\rangle_t \\ &= -Y_t Z_t^{-2}\, dZ_t + Y_t Z_t^{-3} d\langle Z\rangle_t + Z_t^{-1} dY_t - Z^{-2} d\langle Y, Z\rangle_t \\ &= -Y_t Z_t^{-1}\, dM_t + Y_t Z_t^{-1} d\langle M\rangle_t + Z_t^{-1} Y_t dM_t - Y Z^{-1} d\langle M\rangle_t \\ &= 0 \end{aligned}$$

Thus, $Y_t/Z_t = Y_0/Z_0 = 1$ for all $t > 0$, so Y and Z are identical. ∎

The process Z is most useful when it is a martingale. We shall develop a simple sufficient condition under which this is true. As preparation we require the following lemma, which is of independent interest.

Gronwall's Lemma. *Let g and b be non-negative Borel measurable functions defined on $[0,\infty)$ and let a be a non-negative constant. If, for some $t_0 > 0$, we have $\int_0^{t_0} b(s)\, ds < \infty$ and*

(8) $$g(t) \le a + \int_0^t g(s)b(s)\, ds, \qquad \forall t \in [0, t_0],$$

then

(9) $$g(t) \le a \exp\left(\int_0^t b(s)\, ds\right), \qquad \forall t \in [0, t_0].$$

Proof. Define $B(t) := \int_0^t b(s)\, ds$ and $G(t) := \int_0^t g(s)b(s)\, ds$ for $t \in [0, t_0]$. Then

(10) $$\frac{d}{dt}\left[e^{-B(t)} G(t)\right] = e^{-B(t)} b(t)\left[g(t) - G(t)\right] \le a e^{-B(t)} b(t),$$

for a.e. $t \in [0, t_0]$. Integrating the extreme terms in (10) we find that

(11) $$e^{-B(t)} G(t) \le \int_0^t a e^{-B(s)} b(s)\, ds = a\left(1 - e^{-B(t)}\right), \qquad t \in [0, t_0].$$

Thus, $G(t) \le a(e^{B(t)} - 1)$, so

(12) $$g(t) \le a + G(t) \le a e^{B(t)}, \qquad t \in [0, t_0],$$

as claimed. ∎

Theorem 2. *Suppose that $H \in \mathcal{L}^2$ satisfies the bound $|H_s(\omega)| \le f(s)$ for all $(\omega, s) \in \Omega \times [0,\infty)$, where $\int_0^t [f(s)]^2\,ds < \infty$ for each $t > 0$. If Z is the stochastic exponential associated with H as in (1) and (2), then Z is a square-integrable martingale.*

Proof. From Itô's formula,

$$Z_t^2 = 1 + 2\int_0^t Z_s\,dZ_s + \int_0^t Z_s^2 H_s^2\,ds. \tag{13}$$

In particular, $Z_t^2 - \int_0^t Z_s^2 H_s^2\,ds = Z_t^2 - \langle Z\rangle_t$ is a local martingale. Let (T_n^1) be a sequence of stopping times reducing this local martingale. Let (T_n^2) be a sequence of stopping times reducing the local martingale Z. Then $T_n := T_n^1 \wedge T_n^2$ defines a sequence of stopping times that reduces both Z and $Z^2 - \langle Z\rangle$. In particular,

$$\begin{aligned} \mathbf{E}[Z_{t\wedge T_n}^2] &= 1 + \mathbf{E}\left[\int_0^{t\wedge T_n} Z_s^2 H_s^2\,ds\right] \\ &\le 1 + \mathbf{E}\left[\int_0^{t\wedge T_n} Z_s^2 [f(s)]^2\,ds\right] \\ &\le 1 + \mathbf{E}\left[\int_0^{t} Z_{s\wedge T_n}^2 [f(s)]^2\,ds\right] \end{aligned} \tag{14}$$

Let us fix n for a moment and define $g(t) := \mathbf{E}[Z_{t\wedge T_n}^2]$ for $t \in [0,\infty)$. Then (14) implies

$$g(t) \le 1 + \int_0^t g(s)[f(s)]^2\,ds, \qquad t \in [0,\infty). \tag{15}$$

Feeding (15) into Gronwall's lemma we deduce that

$$\mathbf{E}[Z_{t\wedge T_n}^2] \le \exp\left(\int_0^t [f(s)]^2\,ds\right) \qquad t \in [0,\infty), n = 1, 2, \ldots. \tag{16}$$

Now Doob's inequality applied to the stopped process $Z_{t\wedge T_n}$ (a u.i. martingale!) yields

$$\mathbf{E}\left[\sup_{0\le s\le t\wedge T_n} Z_s^2\right] \le 4\mathbf{E}[Z_{t\wedge T_n}^2] \le 4\exp\left(\int_0^t [f(s)]^2\,ds\right), \qquad t \in [0,\infty). \tag{17}$$

It follows from (17) and the "crystal ball" condition that for each $t > 0$ the collection of random variables $\{Z_{s\wedge T_n} : s \in [0,t], n \in \mathbf{N}\}$ is uniformly integrable. In particular, $Z_{t\wedge T_n}$ converges both a.s. and (more importantly) in L^1 to Z_t as $n \to \infty$. Since $(Z_{t\wedge T_n})_{0\le t\le t_0}$ is a martingale, so is its L^1 limit $(Z_t)_{0\le t\le t_0}$. That this martingale is square integrable follows immediately from (16) and Fatou's lemma. ∎

Example 1. (*Cf.* Problem 4, Homework 6): If f is a measurable function on $[0,\infty)$ with $\int_0^t [f(s)]^2\,ds<\infty$ for each $t>0$, then

$$Z_t := \exp\left(\int_0^t f(s)\,dB_s - \tfrac{1}{2}\int_0^t [f(s)]^2\,ds\right), \qquad t\ge 0, \tag{18}$$

is a strictly positive martingale. From this one can deduce, as in the homework problem just cited, that $M_t := \int_0^t f(s)\,dB_s$ is normally distributed with mean 0 and variance $\int_0^t [f(s)]^2\,ds$.

Sharper criteria for Z to be a true martingale are known, but their proofs are more delicate. Let us state the two most well known, without proofs. Notation is as in (1) and (2).

Theorem 3. [Novikov] *If*

$$\mathbf{E}\left[\exp\left(\tfrac{1}{2}\langle M\rangle_t\right)\right]<\infty, \tag{19}$$

then $\mathbf{E}[Z_t]=1$, *in which case* $(Z_s)_{0\le s\le t}$ *is a martingale.*

Theorem 4. [Kazamaki] *If*

$$\sup_{0\le s\le t}\mathbf{E}\left[\exp\left(\tfrac{1}{2}M_s\right)\right]<\infty, \tag{20}$$

then $\mathbf{E}[Z_t]=1$, *in which case* $(Z_s)_{0\le s\le t}$ *is a martingale.*

Remark 1. The form (2) for the local martingale Z may seem very special, but in fact any *strictly positive* local martingale has this form. This is discussed in some detail in the handout on Girsanov's theorem.