In what follows, $X = (X_n)_{n=0}^\infty$ is a Markov chain with countable state space $S$ and transition probability matrix $P = \{p(x,y)\}_{x,y \in S}$. We suppose that $X$ has been constructed on the sequence space $\Omega = S^{\{0,1,2,\ldots\}}$, and that $P_x$ is the probability measure on $\Omega$ corresponding to the initial condition $X_0 = x$. Other notation is that used in class.

We present two criteria, both based on the following observation presented already in class.

**Proposition.** Let $f : S \to [0, \infty)$ satisfy $Pf(x) \leq f(x)$ for all $x \in S \setminus F$, where $F \subset S$. Define a stopping time by $D := \inf\{n \geq 0 : X_n \in F\}$. Then the stopped process $(f(X_{n \wedge D}))_{n \geq 0}$ is a $P_x$-supermartingale for each $x \in S$.

**Proof.** We compute

\[
E_x[f(X_{(n+1) \wedge D})|F_n] = 1_{\{n < D\}}E_x[f(X_{n+1})|F_n] + E_x[f(X_D)1_{\{D \leq n\}}|F_n]
\]

\[
= 1_{\{n < D\}}E_x[f(X_{n+1})|F_n] + f(X_D)1_{\{D \leq n\}}
\]

\[
= 1_{\{n < D\}}Pf(X_n) + f(X_D)1_{\{D \leq n\}}
\]

\[
\leq 1_{\{n < D\}}f(X_n) + f(X_D)1_{\{D \leq n\}}
\]

\[
= f(X_{n \wedge D}).
\]

The inequality in the above computation follows from the hypothesis because $X_n \in S \setminus F$ when $n < D$. These conditional expectation calculations are valid even without knowing that $E_x[f(X_{n \wedge D})]$ is finite, because $f \geq 0$. But having demonstrated the supermartingale inequality, we can now check the required integrability:

\[
E_x[f(X_{n \wedge D})] \leq E_x[f(X_{0 \wedge D})] = f(x) < \infty.
\]

\]

**Remark.** The same proof shows that $f(X_{n \wedge D})$ is a martingale provided $Pf = f$ on $S \setminus F$.

Here is a recurrence criterion for $X$, based on the existence of a (“Liapunov”) function on $S$ with certain properties. This type of criterion seems to have first appeared in the literature in a paper of F.G. Foster [1].
**Theorem 1.** Assume that $X$ is irreducible. Suppose there is a finite set $F \subset S$ and a function $f : S \to [0, +\infty)$ such that

(a) $P_f(x) \leq f(x)$ for all $x \notin F$, and

(b) $\{x \in S : f(x) \leq M\}$ is a finite set for each $M > 0$.

Then the Markov chain $X$ is recurrent.

**Proof.** Define stopping times $D := \inf\{n \geq 0 : X_n \in F\}$ and (for $M \in \mathbb{N}$) $S_M := \inf\{n \geq 0 : f(X_n) > M\}$.

Fix $x \in S$ and suppose that $P_x[S_M = \infty] > 0$ for some $m \in \mathbb{N}$. Notice that $\{S_M = \infty\} \subset \{f(X_n) \leq M\}$ for all $n$. Therefore, there is positive $P_x$ probability that the Markov chain $X$ remains in the finite set $\{x : f(x) \leq M\}$ forever. This in turn implies that, with positive $P_x$ probability, one of the states of $\{x : f(x) \leq M\}$ is visited infinitely often. Such a state must be recurrent; we conclude that every element of $S$ is recurrent because $X$ is irreducible. In short, if $P_x[S_M = \infty] > 0$ for some $x \in S$, then $X$ is recurrent and we are done.

Now suppose that $P_x[S_M = \infty] = 0$ for all $x \in S$ and all $M \in \mathbb{N}$; that is, $P_x[S_M < \infty] = 1$ for all $x \in S$ and all $M \in \mathbb{N}$. From class discussion we know that $f(X_{n\land D}), n \geq 0$, is a non-negative $P_x$-supermartingale for all $x \in S$. By the optional stopping theorem for non-negative supermartingales we therefore have

$$f(x) = E_x[f(X_{0\land D})] \geq E_x[f(X_{S_M\land D})] \geq E_x[f(X_{S_M\land D}); S_M < D] \geq M \cdot P_x[S_M < D],$$

the final inequality following from the fact that $S_M \land D = S_M$ on $\{S_M < D\}$, and thus $f(X_{S_M\land D}) = f(X_{S_M}) \geq M$ on $\{S_M < D\}$. Comparing the extreme terms in (1) we arrive at

$$P_x[S_M < D] \leq f(x)/M, \quad \forall x \in S, \forall M \in \mathbb{N}. \quad (2)$$

Observe that the events $\{S_M < D\}$ are nested inward; that is, $\{S_{M+1} < D\} \subset \{S_M < D\}$ for all $M \in \mathbb{N}$. Letting $M$ tend to $+\infty$ in (2) we therefore obtain

$$P_x[\cap_{M \in \mathbb{N}}\{S_M < D\}] = 0, \quad \forall x \in S. \quad (3)$$

Taking complements:

$$P_x[D \leq S_M \text{ for some } M \in \mathbb{N}] = 1, \quad \forall x \in S. \quad (4)$$
When we combine (4) with the fact that $P_x[S_M < \infty] = 1$ for all $x \in S$ and all $M \in \mathbb{N}$ we obtain

$$P_x[D < \infty] = 1, \quad \forall x \in S. \tag{5}$$

Thus, return to $F$ is certain; by an argument used in class,

$$P_x[X_n \in F \text{ for infinitely many } n] = 1. \tag{6}$$

But $F$ is a finite set, so by the “pigeonhole principle”, (6) implies that there exists $x_0 \in F$ such that

$$P_x[X_n = x_0 \text{ for infinitely many } n] = 1.$$

Of course this state $x_0$ must be recurrent, and so $X$ is recurrent because it is irreducible.

Example 1. Let $\{\xi_n\}_{n \geq 1}$ be an iid sequence of Bernoulli random variables: $P[\xi_n = 1] = P[\xi_n = -1] = 1/2$ for all $n$. Let $b : \mathbb{Z} \to \mathbb{Z}$ satisfy (i) $|b(x)| < |x|$ for all $x \neq 0$, (ii) $b(x) < 0$ for $x > 0$, and (iii) $b(x) > 0$ for $x < 0$. Consider the Markov chain $X = \{X_n\}_{n \geq 0}$ generated recursively by

$$X_{n+1} = X_n + b(X_n) + \xi_{n+1}, \quad n = 0, 1, 2, \ldots.$$

The hypotheses listed above ensure that the “drift” $b(x)$ tends to push $X$ towards the state 0. Thus we expect $X$ to be recurrent. Let us confirm this using Theorem 1. We take $f(x) := |x|$. Then

$$Pf(x) = E_x|X_1| = E_x|x + b(x) + \xi_1| = \frac{1}{2} (|x + b(x) + 1| + |x + b(x) - 1|).$$

Suppose that $x > 0$. Then by (i) and (ii) above we have $0 \leq -b(x) < x$, so $x + b(x) \pm 1 \geq 0$, whence

$$\frac{1}{2} (|x + b(x) + 1| + |x + b(x) - 1|) = x + b(x) < x = |x| = f(x),$$

which verifies that $Pf(x) \leq x$ if $x > 0$. In the same way $Pf(x) \leq f(x)$ if $x < 0$. Therefore Theorem 1 applies to this choice of $f$ with $F = \{0\}$. We conclude that $X$ is recurrent. □
The companion transience criterion is problem 5 on your second homework assignment:

**Theorem 2.** Assume that $X$ is irreducible. Suppose there is a finite set $F$ and a function $g : S \to [0, +\infty)$ such that

(a) $P g(x) \leq g(x)$ for all $x \notin F$, and  
(b) $\inf \{ g(x) : x \in S \} = 0$.

Then $X$ is transient.

**Example 2.** Let us modify Example 1 by now assuming that $b(x) > 0$ if $x > 0$ and $b(x) < 0$ if $x < 0$. The drift $b(x)$ is now driving $X$ away from 0, so we expect $X$ to be transient. Use Theorem 2 to verify this intuition. □

An excellent source for results of the type presented here (and much more) is the book of Meyn and Tweedie [2].

**Reference**
