Girsanov’s Theorem

In what follows, \((\Omega, \mathcal{F}, \mathbb{P})\) is the canonical sample space of the Brownian motion \((B_t)_{t \geq 0}\) with \(B_0 = 0\); other notation is that used in class.

Let \(Q\) be a second probability measure on \((\Omega, \mathcal{F})\) that is locally mutually absolutely continuous with respect to \(P\) (in symbols \(Q \sim P\)), in the sense that that \(Q(A) = 0\) if and only if \(P(A) = 0\), for all \(A \in \mathcal{F}_t\) and all \(t > 0\). Then the restrictions \(Q^t\) and \(P^t\), of \(Q\) and \(P\) to \(\mathcal{F}_t\), are likewise mutually absolutely continuous, and the process of Radon-Nikodym derivatives

\[
Z_t := \frac{dQ^t}{dP^t}, \quad t \geq 0,
\]

is a \(P\) martingale. We can (and do) choose a modification of \(Z\) that is right continuous. In fact, by the martingale representation theorem, the process \(Z\) has continuous paths. Notice that \(E^P[Z_0] = 1\), while Blumenthal’s zero-one law implies that \(\mathcal{F}_{0+}\) is \(P\)-trivial, hence \(Q\)-trivial. Therefore \(P[Z_0 = 1] = Q[Z_0 = 1] = 1\).

Just as in discrete time, a positive martingale “sticks” at zero once it attains that value. But \(Q[Z_t = 0] = \int_{\{Z_t = 0\}} Z_t dP = 0\), so \(Z_t > 0\), \(Q\)-a.s., hence \(P\)-a.s. It follows from this and path continuity that

\[
\inf_{0 \leq s \leq t} Z_s(\omega) > 0, \quad \forall t > 0, \text{ for } P\text{-a.e. } \omega.
\]

This implies that the stochastic integral

\[
M_t := \int_0^t Z_s^{-1} dZ_s, \quad t \geq 0,
\]

exists and defines a continuous local martingale \((M_t)_{t \geq 0}\). By the martingale representation theorem, there exists \(H \in \mathcal{L}^2_{loc}\) such that \(M = H \cdot B\). We then have,

\[
Z_t = 1 + \int_0^t Z_s dM_s = 1 + \int_0^t Z_s H_s dB_s, \quad t \geq 0,
\]

so by the uniqueness of the stochastic exponential as solution of such an equation,

\[
Z_t = \exp \left( M_t - \frac{1}{2} \langle M \rangle_t \right), \quad \forall t \geq 0, P\text{-a.s.}
\]

Now suppose that \(F \in b\mathcal{F}_s\) and \(G \in \mathcal{F}_t\), where \(0 \leq s < t\). Then

\[
E^Q[FG] = E^P[FGZ_t] = E^P[FE^P[GZ_t|\mathcal{F}_s]]
\]

\[
= E^P \left[ F \frac{E^P[GZ_t|\mathcal{F}_s]}{E^P[Z_t|\mathcal{F}_s]} Z_s \right]
\]

\[
= E^Q \left[ F \frac{E^P[GZ_t|\mathcal{F}_s]}{E^P[Z_t|\mathcal{F}_s]} \right].
\]
It follows that Q conditional expectations can be expressed in terms of P conditional expectations by the formula

$$
E^Q[G|\mathcal{F}_s] = \frac{E^P[GZ_t|\mathcal{F}_s]}{E^P[Z_t|\mathcal{F}_s]}, \quad G \in b\mathcal{F}_t, 0 \leq s < t.
$$

Next let \( N = (N_t)_{t \geq 0} \) be an adapted process. Then

$$
E^Q[|N_t|] = E^P[|N_tZ_t|],
$$

so \( N_t \) is Q-integrable if and only if \( N_tZ_t \) is P-integrable. If things are so, then by (1)

$$
E^Q[N_t|\mathcal{F}_s] = \frac{E^P[N_tZ_t|\mathcal{F}_s]}{Z_s},
$$

which is equal to \( N_s \) (a.s.) if and only if \( E^P[N_tZ_t|\mathcal{F}_s] = N_sZ_s \). In other words, \( N \) is a Q-martingale if and only if \( NZ \) is a P-martingale. In particular, every right continuous Q-martingale is, in fact, continuous.

Consider now an Itô process

$$
X_t = X_0 + \int_0^t K_s dB_s + \int_0^t v_s ds = Y_t + V_t.
$$

Our aim is to describe in concrete terms the circumstances under which \( X \) is a Q local martingale. By the preceding paragraph and a localization argument, \( X \) is a Q local martingale if and only if \( XZ \) is a P local martingale. But, by Itô’s formula,

$$
X_tZ_t = X_0 + \int_0^t X_s dZ_s + \int_0^t Z_s dX_s + \langle X, Z \rangle_t
$$

$$
= X_0 + \int_0^t X_s dZ_s + \int_0^t Z_s K_s dB_s + \int_0^t Z_s v_s ds + \int_0^t K_s H_s Z_s ds
$$

$$
= X_0 + \int_0^t X_s dZ_s + \int_0^t Z_s K_s dB_s + \int_0^t (v_s + K_s H_s) Z_s ds.
$$

It follows that \( XZ \) is a P local martingale if and only if \( (v_s + K_s H_s)Z_s = 0 \) for a.e. \((\omega, s) \in \Omega \times [0, \infty)\). But \( Z > 0 \), so this condition holds if and only if \( v = -HK \) a.e. on \( \Omega \times [0, \infty) \). Another way of saying this is that for any \( K \in \mathcal{L}^2_{loc} \), the process

$$
X_t = Y_t - \langle Y, M \rangle_t
$$

is a Q local martingale, where \( Y = K \bullet B \) and \( M = H \bullet B \) are P local martingales, as above. The moral of the story is that when we change the governing measure from P to Q we must compensate by subtracting the term \( \langle Y, M \rangle \) to turn the P local martingale \( Y \) into a Q-local martingale.

The above discussion can be summarized as follows.
Cameron-Martin-Girsanov Theorem. (a) If $Q$ is a probability measure on $(\Omega, \mathcal{F})$ that is locally mutually absolutely continuous with respect to $P$, then there is a local martingale $M = H \cdot B$ ($H \in L^2_{\text{loc}}$) such that the Radon-Nikodym martingale $Z_t := dQ(t)/dP(t)$ is equal to the exponential martingale $\exp(M_t - \frac{1}{2}\langle M \rangle_t)$. Moreover, if $N$ is a $P$ local martingale, then $\tilde{N} := N - \langle N, M \rangle$ is a $Q$ local martingale and $\langle \tilde{N} \rangle = \langle N \rangle$, almost surely.

(b) Conversely, given a continuous local martingale $M$, define the exponential local martingale $Z$ as above. If $Z$ is a true martingale, then there exists a unique probability measure $Q$ on $(\Omega, \mathcal{F})$ such that $Q(t) = Z_t \cdot P(t)$ for each $t > 0$.

The final assertion in part (a) follows from the sum-of-squares approximation of the quadratic variation process $\langle N \rangle$; note that because $P \sim Q$, the notions “convergence in $Q$ probability” and “convergence in $P$ probability” are the same.

The case of constant $H$, say $H_s(\omega) = \mu$, is especially interesting. In this case $Z$ is the familiar exponential martingale

$$Z_t = \exp(\mu B_t - \mu^2 t/2).$$

By the above discussion,

$$\beta_t := B_t - \mu t$$

is a $Q$ local martingale. One of your final homework problems involves showing that $(\beta_t)_{t \geq 0}$ is in fact a Brownian motion under $Q$. Granted this, we see that

$$B_t = \beta_t + \mu t,$$

so that the $Q$ distribution of $B$ is that of a Brownian motion $(\beta_t)$ plus a drift $\mu t$.

More generally, let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel measurable function. Then by results proved in class,

$$Z_t := \exp \left( \int_0^t f(B_s) dB_s - \frac{1}{2} \int_0^t [f(B_s)]^2 ds \right)$$

is a positive martingale. Let $Q$ be the associated probability measure:

$$Q(A) = \int_A Z_t(\omega) P(d\omega), \quad A \in \mathcal{F}_t, t \geq 0.$$

By the preceding discussion and Levy’s theorem, one can show that, under $Q$,

$$\beta_t := B_t - \int_0^t f(B_s) ds$$

is standard Brownian motion. Therefore the canonical process $B$ on the probability space $(\Omega, \mathcal{F}, Q)$ solves the “stochastic differential equation”

$$dX_t = d\beta_t + f(X_t) dt.\quad (2)$$