

Math 280C, Spring 2005

Lévy's Theorem

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space endowed with a right-continuous* filtration $(\mathcal{F}_t)_{t \geq 0}$ such that \mathcal{F}_0 contains all the \mathbf{P} -null sets in \mathcal{F} and $\bigvee_t \mathcal{F}_t = \mathcal{F}$. Let $M = (M_t)_{t \geq 0}$ be a real-valued stochastic process adapted to (\mathcal{F}_t) with continuous sample paths. We assume that $M_0 = 0$.

Theorem. *Suppose that both M and $(M_t^2 - t)_{t \geq 0}$ are local martingales. Then M is a Brownian motion with respect to (\mathcal{F}_t) . More precisely, if $0 < s < t$, then $M_t - M_s$ is independent of \mathcal{F}_s and is normally distributed with mean 0 and variance $t - s$.*

Proof. The key observation (due to H. Kunita & S. Watanabe) is that the development of the Itô integral (and Itô's formula) for Brownian motion B_t rests solely on the fact that B_t and $B_t^2 - t$ are (local) martingales. It follows that if $f \in C^2(\mathbf{R})$ then

$$(1) \quad f(M_t) = f(0) + \int_0^t f'(M_s) dM_s + \frac{1}{2} \int_0^t f''(M_s) ds,$$

where the stochastic integral $M_t^f := \int_0^t f'(M_s) dM_s$ is a local martingale. In particular, if f and its derivatives f' and f'' are bounded, then M_t^f is a martingale, in which case upon taking expectations in (1) we obtain

$$(2) \quad \mathbf{E}[f(M_t)] = f(0) + \frac{1}{2} \int_0^t \mathbf{E}[f''(M_s)] ds.$$

Let us take f in (2) to be of the form $f(x) = \exp(i\theta x)$, where $\theta \in \mathbf{R}$ and $i = \sqrt{-1}$. Writing $g(t) := \mathbf{E}[\exp(i\theta M_t)]$ we obtain

$$g(t) = 1 - \frac{\theta^2}{2} \int_0^t g(s) ds$$

because $f''(x) = -\theta^2 f(x)$. Consequently, g satisfies the initial value problem

$$g'(t) = -\frac{\theta^2}{2} g(t) \quad g(0) = 1,$$

which has the unique solution $g(t) = \exp(-t\theta^2/2)$. Thus

$$\mathbf{E}[\exp(i\theta M_t)] = \exp(-t\theta^2/2), \quad \theta \in \mathbf{R},$$

* i.e., $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \geq 0$.

which means that $M_t \sim \mathcal{N}(0, t)$.

Now fix $s > 0$ and $A \in \mathcal{F}_s$ with $\mathbf{P}(A) > 0$. Define $\mathbf{P}^*(B) := \mathbf{P}(B \cap A)/\mathbf{P}(A) = \mathbf{P}(B|A)$, $\mathcal{F}_t^* := \mathcal{F}_{t+s}$, and $M_t^* := M_{t+s} - M_s$ for $t \geq 0$. Then with respect to the filtration (\mathcal{F}_t^*) over the probability space $(\Omega, \mathcal{F}, \mathbf{P}^*)$, the stochastic process $(M_t^*)_{t \geq 0}$ is a continuous local martingale with $M_0^* = 0$ such that $[M_t^*]^2 - t$ is also a local martingale. The considerations of the preceding paragraph apply to this process, and we deduce that

$$(3) \quad \mathbf{E}^*[\exp(i\theta M_t^*)] = \exp(-t\theta^2/2).$$

Writing the “starred” objects explicitly, (3) becomes

$$(4) \quad \mathbf{E}[\exp(i\theta(M_{t+s} - M_t)); A] = \exp(-t\theta^2/2)\mathbf{P}(A).$$

Varying $A \in \mathcal{F}_s$ in (4) we find that

$$\mathbf{E}[\exp(i\theta(M_{t+s} - M_t))|\mathcal{F}_s] = \exp(-t\theta^2/2),$$

which shows that $M_{t+s} - M_s$ is independent of \mathcal{F}_s and has the $\mathcal{N}(0, t)$ distribution. \square

Example 1. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a filtered probability space, and let $B = (B_t)_{t \geq 0}$ be an (\mathcal{F}_t) Brownian motion. Let $H = (H_t)_{t \geq 0}$ be a measurable (\mathcal{F}_t) adapted process taking on only the two values ± 1 . Then $H \in \mathcal{L}^2$ so the stochastic integral

$$M_t := \int_0^t H_s dB_s, \quad t \geq 0,$$

is a square-integrable martingale. Moreover, $\langle M \rangle_t = \int_0^t H_s^2 ds = \int_0^t 1 ds = t$, so $M_t^2 - t$ is also a local martingale. It follows from Lévy’s theorem that M is also an (\mathcal{F}_t) Brownian motion.

Example 2. As in the previous example, let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a filtered probability space, and let $B = (B_t)_{t \geq 0}$ be an (\mathcal{F}_t) Brownian motion. Now let $H = (H_t)_{t \geq 0}$ be an arbitrary element of $\mathcal{L}_{\text{loc}}^2$. As before we define the local martingale

$$M_t := \int_0^t H_s dB_s, \quad t \geq 0,$$

which has quadratic variation process

$$\langle M \rangle_t = \int_0^t H_s^2 ds, \quad t \geq 0.$$

Observe that $\langle M \rangle_t$ is continuous and non-decreasing. Let us assume that, almost surely,

$$(5) \quad \lim_{t \rightarrow \infty} \langle M \rangle_t = \infty.$$

Define

$$T(s) := \inf\{t : \langle M \rangle_t > s\}, \quad s \geq 0.$$

Then (5) implies that $T(s)$ is finite (a.s.) for each $s \geq 0$, and it is not hard to check that each $T(s)$ is a stopping time. We use these stopping times to “time change” M into Brownian motion. Precisely, define

$$C_s := M_{T(s)}, \quad s \geq 0,$$

and

$$\mathcal{G}_s := \mathcal{F}_{T(s)}, \quad s \geq 0.$$

Then the stochastic process $C = (C_s)_{s \geq 0}$ is adapted to the filtration $(\mathcal{G}_s)_{s \geq 0}$. The optional stopping theorem implies that C is a local martingale (with respect to (\mathcal{G}_s)). Moreover, the quadratic variation interpretation of $\langle C \rangle$ and $\langle M \rangle$ implies that

$$\langle C \rangle_s = \langle M \rangle_{T(s)} = s, \quad \forall s \geq 0,$$

almost surely. That is, $C_s^2 - s$ is also a (\mathcal{G}_s) local martingale. Lévy’s theorem now tells us that $C = (C_s)$ is a (\mathcal{G}_s) Brownian motion. The moral: A continuous local martingale is just Brownian motion with its “clock” running too fast (or too slow).