Math 280C, Spring 2005

Stochastic Integral

In what follows, $(\Omega, \mathcal{F}, \mathbf{P})$is the canonical sample space of the Brownian motion $(B_t)_{t\geq 0}$ with $B_0 = 0$; other notation is that used in class. We define the stochastic integral with respect to Brownian motion in several stages.

1. Suppose $0 \leq u < v$ and $G \in L^2(\mathcal{F}_u)$. Define $H_s(\omega) := G(\omega)1_{(u,v]}(s)$. It is then natural to define

$$(H \bullet B)_t = \int_0^t H_s \, dB_s := G \cdot (B_{v\wedge t} - B_{u\wedge t}), \qquad t \geq 0. \tag{1.1}$$

Observe that $H \bullet B$ is a (path) continuous martingale with initial value equal to 0. Moreover, because G is independent of $B_{v\wedge t} - B_{u\wedge t}$,

$$\begin{aligned} \mathbf{E}[(H \bullet B)_t^2] &= \mathbf{E}[G^2 \cdot (B_{v\wedge t} - B_{u\wedge t})^2] \\ &= \mathbf{E}[G^2] \cdot \mathbf{E}[(B_{v\wedge t} - B_{u\wedge t})^2] \\ &= \mathbf{E}[G^2] \cdot ((v \wedge t) - (u \wedge t)) \\ &= \mathbf{E}[G^2] \int_0^t 1_{(u,v]}(s) \, ds \\ &= \mathbf{E} \int_0^t H_s^2 \, ds. \end{aligned} \tag{1.2}$$

Thus $H \bullet B$ is even a *square-integrable* martingale: $\mathbf{E}[(H \bullet B)_t^2] < \infty$ for each $t \geq 0$. We use $\mathcal{M}^2$ to denote the class of square-integrable martingales with continuous sample paths and initial value 0. By the preceding discussion, $H \bullet B \in \mathcal{M}^2$.

If now we have two integrands $H^{(i)} = G_i 1_{(u_i,v_i]}$, $i = 1, 2$, of the above form, then a straightforward calculation shows that

$$\mathbf{E}[(H^{(1)} \bullet B)_t \cdot (H^{(2)} \bullet B)_t] = \mathbf{E}\left[\int_0^t H_s^{(1)} H_s^{(2)} \, ds\right], \qquad t \geq 0. \tag{1.3}$$

2. Consider an integrand H that is a sum of integrands of the type discussed above:

$$H_s(\omega) = \sum_{k=1}^n G_k(\omega) \cdot 1_{(u_k,v_k]}(s), \qquad \omega \in \Omega, s \geq 0, \tag{2.1}$$

where $n \in \mathbf{N}$, $0 \leq u_k < v_k$, and $G_k \in L^2(\mathcal{F}_{u_k})$. We let $\mathcal{L}_e^2$ denote the class of all such integrands. Notice that each $H \in \mathcal{L}_e^2$ satisfies (i) $s \mapsto H_s(\omega)$ is a left-continuous step function for each $\omega \in \Omega$, (ii) $H_s \in L^2(\mathcal{F}_s)$ for each $s \geq 0$, (iii) viewed as a mapping from $\Omega \times [0, t]$ to $\mathbf{R}$, $(\omega, s) \mapsto H_s(\omega)$ is $\mathcal{F}_t \otimes \mathcal{B}_{[0,t]}$ measurable for each $t > 0$, and (iv) $\mathbf{E}[\int_0^t H_s^2 \, ds] < \infty$ for each $t > 0$. We now define, for $H \in \mathcal{L}_e^2$ as displayed in (2.1),

$$(H \bullet B)_t = \int_0^t H_s \, dB_s := \sum_{k=1}^n G_k \cdot (B_{v_k\wedge t} - B_{u_k\wedge t}), \qquad t \geq 0. \tag{2.2}$$

By the discussion in **1**, $H \bullet B$ is an element of $\mathcal{M}^2$ for each $H \in \mathcal{L}_e^2$. In fact, the mapping $I : H \mapsto H \bullet B$ is a linear mapping of $\mathcal{L}_e^2$ into $\mathcal{M}^2$ that preserves norms and inner products; indeed, by (1.3) and linearity of integration, we have the "Itô isometry":

$$\mathbf{E}[(H \bullet B_t) \cdot (K \bullet B)_t] = \mathbf{E} \int_0^t H_s K_s \, ds = \int_0^t \mathbf{E}[H_s \cdot K_s] \, ds, \tag{2.3}$$

for each $t \geq 0$, and $H, K \in \mathcal{L}_e^2$. In particular,

$$\mathbf{E}[(H \bullet B)_t^2] = \mathbf{E} \int_0^t H_s^2 \, ds = \int_0^t \mathbf{E}[H_s^2] \, ds, \tag{2.3}$$

for each $t \geq 0$, and $H \in \mathcal{L}_e^2$.

The extension of the integral defined above to more general integrands is based on (2.3), Doob's inequality, and the following lemma. Let us now define $\mathcal{L}^2$ to be the class of all integrands $H = H_s(\omega)$ satisfying the following conditions: (i) as a mapping from $\Omega \times [0,t]$ to $\mathbf{R}$, $(\omega, s) \mapsto H_s(\omega)$ is $\mathcal{F}_t \otimes \mathcal{B}_{[0,t]}$ measurable for each $t > 0$, and (ii) $\mathbf{E}[\int_0^t H_s^2 \, ds] < \infty$ for each $t > 0$. In particular, H_s is $\mathcal{F}_s$ measurable for each $s \geq 0$.

3. Lemma. *Given $H \in \mathcal{L}^2$ there is a sequence $(H^{(n)}) \subset \mathcal{L}_e^2$ such that*

$$\mathbf{E}\left[\int_0^t (H_s - H_s^{(n)})^2 \, ds\right] \to 0 \tag{3.1}$$

as $n \to \infty$.

Proof. Fix $t > 0$ and define $\mathcal{L}_e^2(t) := \{H|_{\Omega \times [0,t]} : H \in \mathcal{L}_e^2\}$ and $\mathcal{L}^2(t) := \{H|_{\Omega \times [0,t]} : H \in \mathcal{L}^2\}$. We need to show that $\mathcal{L}_e^2(t)$ is dense in $\mathcal{L}^2(t)$ with respect to the L^2-norm on $\Omega \times [0,t]$. Let $\mathcal{K} := \{K \in \mathcal{L}^2(t) : \mathbf{E}[\int_0^t H_s \cdot K_s \, ds] = 0, \forall H \in \mathcal{L}_e^2(t)\}$, the orthogonal complement of $\mathcal{L}_e^2(t)$ in $\mathcal{L}^2(t)$. By Hilbert space theory, the orthogonal complement of $\mathcal{K}$ (in $\mathcal{L}^2(t)$) is equal to the closure of $\mathcal{L}_e^2(t)$ in $\mathcal{L}^2(t)$. We show that $\mathcal{K} = \{0\}$ (the trivial subspace consisting of only the zero random variable). From this it will follow immediately that the complement of $\mathcal{K}$ is all of $\mathcal{L}^2(t)$, and we will be done.

So suppose that $K \in \mathcal{K}$. Fix u and v with $0 < u < v < t$ and $G \in L^2(\mathcal{F}_u)$. For $\delta \in (0, t-v)$ define $H^{(\delta)} \in \mathcal{L}_e^2(t)$ by

$$H_s^{(\delta)}(\omega) := G(\omega) \cdot 1_{(v, v+\delta]}(s), \qquad \omega \in \Omega, s \in [0,t].$$

Then, because $K \in \mathcal{K}$,

$$0 = \mathbf{E}\left[\int_0^t K_s H_s^{(\delta)} \, ds\right] = \int_v^{v+\delta} \mathbf{E}\left[K_s \cdot G\right] \, ds. \tag{3.2}$$

Divide both sides of (3.2) by δ and then let δ fall to 0. By real analysis we obtain, at least for almost every $v \in (u,t)$,

$$0 = \mathbf{E}[K_v \cdot G].$$

Varying $u \in (0, v)$ and G over a suitable countable π-system generating $\mathcal{F}_v$, we find that for (Lebesgue) almost every $v \in (0, t)$,

$$\mathbf{E}[K_v \cdot G] = 0, \qquad \forall G \in L^2(\mathcal{F}_v).$$

Taking $G = K_v$ in (3.3) we see that

$$\mathbf{E}[K_v^2] = 0$$

for almost every $v \in (0, t)$. That is, by Fubini, $K_v(\omega) = 0$ for $\mathbf{P} \otimes$ Lebesgue a.e. $(\omega, v) \in \Omega \times [0, t]$. This proves that $\mathcal{K}$ consists solely of the zero function. □

The following form of Doob's inequality is a simple consequence of the discrete time result discussed in 280B; see the 280B handout on Doob's inequalities.

4. Theorem. *If M is an element of $\mathcal{M}^2$, then*

$$\mathbf{E}\left[\sup_{0 \le s \le t} M_s^2\right] \le 4\mathbf{E}[M_t^2] \tag{4.1}$$

for each $t > 0$.

Proof. Fix $t > 0$. For each positive integer n, the discrete time process $M_{k2^{-n}t}$, $k = 0, 1, 2, \dots, 2^n$, is a square-integrable martingale. By the discrete time Doob inequality,

$$\mathbf{E}\left[\sup_{k=0,1,2,\dots,2^n} M_{k2^{-n}t}^2\right] \le 4\mathbf{E}[M_t^2]. \tag{4.2}$$

The path continuity of M implies that $\sup_{k=0,1,2,\dots,2^n} M_{k2^{-n}t}^2$ increases pointwise to $\sup_{0 \le s \le t} M_s^2$ as $n \to \infty$. Thus (4.1) follows from (4.2) and the monotone convergence theorem. □

5. Theorem. *The linear map $I : H \mapsto H \bullet B$ (defined above for $H \in \mathcal{L}_e^2$) extends uniquely to a continuous linear map of $\mathcal{L}^2$ into $\mathcal{M}^2$, still denoted by $I(H) = H \bullet B$. We have*

$$\mathbf{E}[(H \bullet B)_t] = \mathbf{E}\int_0^t H_s^2\, ds, \qquad \forall t \ge 0, H \in \mathcal{L}^2. \tag{5.1}$$

Moreover, if $H \in \mathcal{L}^2$ and if T is a stopping time then $1_{(0,T]}H \in \mathcal{L}^2$ and

$$I_{t \wedge T} = I(1_{(0,T]}H)_t, \qquad \forall t \ge 0, \quad \mathbf{P}\text{-a.s.} \tag{5.2}$$

Proof. Given $H \in \mathcal{L}^2$, Lemma 3 guarantees the existence of a sequence (H^n) from $\mathcal{L}_e^2$ converging to H in the sense that

$$\mathbf{E}\int_0^t (H_s - H_s^n)^2\, ds \to 0 \text{ as } n \to \infty \quad \forall t > 0. \tag{5.3}$$

In view of (5.1), the sequence $I(H^n)_t$ is a Cauchy sequence for each $t > 0$. Thus, for each $t > 0$ there is a random variable $I(H)_t$ such that $I(H^n)_t \to I(H)_t$ in L^2. In view of Theorem 4, the convergence is even uniform on compact time intervals; that is,

$$\begin{aligned}\mathbf{E}\left[\sup_{0\le s\le t}|I(H^n)_s - I(H^m)_s|^2\right] &\le 4\mathbf{E}|I(H^n)_t - I(H^m)_t|^2\\ &= 4\int_0^t (H^n_s - H^m_s)^2\,ds \to 0 \text{ as } m,n\to\infty,\end{aligned}$$

and so (Chebyshev's inequality)

$$\sup_{0\le s\le t}|I(H^n)_s - I(H^m)_s| \xrightarrow{P} 0, \tag{5.4}$$

for each $t > 0$. Because the uniform limit of continuous functions is uniform, it follows that $s \mapsto I(H)_s$ can be chosen to be continuous, $\mathbf{P}$-a.s. Because conditional expectation contracts the L^2 norm, we have (the limits below being in the sense of L^2)

$$\mathbf{E}[I(H)_t|\mathcal{F}_s] = \mathbf{E}[\lim_n I(H^n)_t|\mathcal{F}_s] = \lim_n \mathbf{E}[I(H^n)_t|\mathcal{F}_s] = \lim_n I(H^n)_s = I(H)_t,$$

proving the martingale property of $I(H)$. The "Itô isometry" clearly persists because the convergence of $I(H^n)$ to $I(H)$ is in L^2. We omit the proof of (5.2), but notice that it follows immediately from Corollary 7 below in the case of left continuous integrands. ∎

6. Notation. $\int_0^t H_s\,dB_s := I(H)_t = (H \bullet B)_t$.

7. Corollary. *If $H \in \mathcal{L}^2$ and $s \mapsto H_s(\omega)$ is left continuous, then*

$$\int_0^t H_s\,dB_s = \lim_n \sum_{k=0}^{n-1} H_{kt/n}(B_{(k+1)t/n} - B_{kt/n})$$

the limit being in the sense of convergence in probability.

Proof. Fix $t > 0$. Left continuity of H implies that if

$$H^n_s := \sum_{k=0}^{n-1} H_{kt/n}\cdot 1_{(kt/n,(k+1)t/n]}$$

converges to H in $\mathcal{L}^2_{[0,t]}$. ∎

8. Theorem. [Product Rule] *If H and K are elements of $\mathcal{L}^2$, and $M := H\bullet B$, $N := K\bullet B$, then*

$$M_tN_t = \int_0^t (M_sK_s + N_sH_s)\,dB_s + \int_0^t H_sK_s, \qquad \forall t>0, \mathbf{P}\text{-a.s.}, \tag{8.1}$$

and the stochastic integral on the right side of (8.1) is a martingale.

Proof. The assertion follows for H and K in $\mathcal{L}^2_e$ by a straightforward (but rather tedious) calculation. The general case then follows by approximation—notice that if $H^n \to H$ and $K^n \to K$ in $\mathcal{L}^2$, then $I(H^n)_t \cdot I(K^n)_t \to M_tN_t$ in L^1 for each $t > 0$. ∎

9. Local martingale. An adapted process $M = (M_t)_{t\ge 0}$ is a *continuous local martingale* provided there is an increasing sequence $(T_n)_{n\ge 0}$ of stopping times with $\lim_n T_n = +\infty$, $\mathbf{P}$-a.s., such that the stopped process $M_t^{T_n} := M_{t\wedge T_n}$ is a uniformly integrable martingale for each n. We say that M is reduced by the sequence (T_n). We let $\mathcal{M}_{\rm loc}$ denote the class of continuous local martingales that vanish at time 0.

10. Lemma. *Given $M \in \mathcal{M}_{\rm loc}$ define*

$$S_k := \inf\{t : |M_t| > k\}, \qquad k = 1, 2, \ldots, \tag{10.1}$$

Then M is reduced by (S_k).

Proof. It is clear that S_k increases to $+\infty$ as $k \to \infty$. Let (T_n) be any reducing sequence for M. Then for each k and n, $t \mapsto M_{t\wedge S_k\wedge T_n}$ is a bounded (by k) martingale. Therefore

$$\mathbf{E}[M_{t\wedge S_k\wedge T_n}|\mathcal{F}_s] = M_{s\wedge S_k\wedge T_n} \tag{10.2}$$

if $0 \le s < t$. We can now send n to infinity in (10.2), making use of the dominated convergence theorem. There results

$$\mathbf{E}[M_{t\wedge S_k}|\mathcal{F}_s] = M_{s\wedge S_k},$$

which proves that M^{S_k} is a bounded (hence u.i.) martingale. ☐

11. Quadratic variation. It can be shown that if M and N are elements of $\mathcal{M}_{\rm loc}$, there is a continuous adapted process $\langle M, N\rangle$ with paths of bounded variation such that

$$\langle M, N\rangle_t = \lim_n \sum_{k=0}^{n-1} [M_{(k+1)t/n} - M_{kt/n}] \cdot [N_{(k+1)t/n} - N_{kt/n}] \tag{11.1}$$

in probability, for each $t > 0$. The process $\langle M, N\rangle$ is called the (quadratic) covariation process associated with M and N; when $M = N$ we write $\langle M\rangle$ instead of $\langle M, M\rangle$ and refer to the quadratic variation of M. We have

$$M_tN_t - \langle M, N\rangle_t$$

is a continuous local martingale. From this, (8.1), and a localization argument it can be see that if $M = H \bullet B$ and $N = K \bullet B$ then

$$\langle M, N\rangle_t = \int_0^t H_sK_s\,ds, \qquad \forall t \ge 0, \tag{11.2}$$

$\mathbf{P}$-a.s. As we shall see, every local martingale of the filtration of our Brownian motion is of the form $H \bullet B$ for some $H \in \mathcal{L}^2_{\rm loc}$, so formula (11.2) is quite general.

12. Notation. We now localize the class of integrands for use in the Itô integral by defining

$$\mathcal{L}^2_{\rm loc} := \{H : H \text{ is jointly measurable and adapted, and } \int_0^t H_s^2\,ds < \infty, \forall t > 0, \mathbf{P}\text{-a.s.}\}.$$

For example, if H is an adapted process such that $s \mapsto H_s(\omega)$ is continuous (or merely right-continuous with left limits) for each ω, then $H \in \mathcal{L}^2_{\rm loc}$. Most every integrand H we shall encounter is of this type.

13. Theorem. *The Itô integral of Theorem 5 extends to a linear map $H \mapsto H \bullet B$ from $\mathcal{L}^2_{\rm loc}$ into $\mathcal{M}_{\rm loc}$. As such, the local martingale $M := H \bullet B$ is uniquely determined by the fact that*

$$\langle M, L\rangle_t = \int_0^t H_s \, d\langle B, L\rangle_s, \qquad \forall t \ge 0,$$

$\mathbf{P}$*-a.s., for each $L \in \mathcal{M}^2$.*

Proof. Fix $H \in \mathcal{L}^2_{\rm loc}$ and define $T_n := \inf\{t : \int_0^t H_s^2 \, ds > n\}$. Clearly (T_n) is an increasing sequence of stopping times that converges to $+\infty$ almost surely. Now define $H_s^{(n)} := 1_{(),T_n]}(s) H_s$. Because $\int_0^t [H_s^{(n)}]^2 \, ds \le \int_0^{t\wedge T_n} H_s^2 \, ds \le n$, each $H^{(n)}$ is an element of $\mathcal{L}^2$. Let $M^{(n)} := I(H^{(n)})$ denote the associated element of $\mathcal{M}^2$. Using (5.2) we compute, for $m < n$ and $t \in [0, T_m]$,

$$\begin{aligned} M_t^{(n)} &= M_{t\wedge T_m}^{(n)} = I(1_{(0,T_m]} H^{(n)})_t \\ &= I(1_{(0,T_m]} 1_{(0,T_n]} H)_t = I(1_{(0,T_m]} H)_t = M_t^{(m)}. \end{aligned} \tag{13.1}$$

Thus the limit

$$M_t := \lim_n M_t^{(n)}, \qquad t \ge 0,$$

is $\mathbf{P}$-a.s. well defined and

$$M_{t\wedge T_n} = M_t^{(n)}, \qquad t \ge 0,$$

which is a bounded martingale. It follows that M is a local martingale reduced by (T_n). We now *define*

$$I(H)_t = (H \bullet B)_t = \int_0^t H_s \, dB_s := M_t. \tag{13.2}$$

Suppose N is a second element of $\mathcal{M}_{\rm loc}$ such that $\langle N, L\rangle_t = \int_0^t H_s \, d\langle B, L\rangle_s$ for all $t \ge 0$ and each $L \in \mathcal{M}^2$. Subtracting we obtain

$$\langle M - N, L\rangle_t = 0, \qquad \forall t \ge 0,$$

$\mathbf{P}$-a.s. Define $R_n := \inf\{t : |M_t| > n\}$ and $S_n := \inf\{t : |N_t| > n\}$. Then (R_n) reduces N, (S_n) reduces M, and (T_n) defined by $T_n := R_n \wedge S_n$, reduces both M and N (to bounded martingales). In particular, if $L^{(n)}$ is the martingale $(M-N)^{T_n}$ obtained by stopping the difference $M - N$ at time T_n, then $L^{(n)}$ is a bounded martingale, hence an element of $\mathcal{M}^2$. Therefore,

$$0 = \langle M - N, L^{(n)}\rangle_t = \langle M - N, M - N\rangle_t, \qquad 0 \le t \le T_n. \tag{13.3}$$

the second equality following from the definition **11** because $L^{(n)} = M - N$ on $[0, T_n]$. Varying n we find that $\langle M - N\rangle_t = 0$ for all $t \ge 0$, $\mathbf{P}$-a.s. In view of Lemma 14 to follow, this means that $M - N = 0$ (the zero process), thus proving the uniqueness assertion.

14. Lemma. *Suppose that $M \in \mathcal{M}_{\rm loc}$ and that $\langle M\rangle_t = 0$ for all $t \ge 0$, $\mathbf{P}$-a.s. Then $\mathbf{P}[M_t = 0, \forall t \ge 0] = 1$.*

Proof. By the discussion of **11**, $L_t := M_t^2 - \langle M\rangle_t$ is a continuous local martingale. Let (T_n) reduce $\mathbf{L}$. Then $L_0 = 0$ so for $n \in \mathbf{N}$ and $t \ge 0$

$$0 = \mathbf{E}[L_0] = \mathbf{E}[M^2_{t\wedge T_n} - \langle M\rangle_{t\wedge T_n}] = \mathbf{E}[M^2_{t\wedge T_n}].$$

Therefore, by Fatou,

$$\mathbf{E}[M_t^2] = 0, \qquad \forall t \ge 0.$$

Consequently, $\mathbf{P}[M_t = 0] = 1$ for each $t \ge 0$. It follows that with probability one, M vanishes at every rational time, and then by path continuity at every time. ∎

15. Corollary. *If $M \in \mathbf{M}$ has paths of bounded variation, then $M = 0$.*

Proof. Fix $t > 0$. Let $V_t := \sup_\pi \sum_k |M_{t_{k+1}} - M_{t_k}|$ (the supremum extending over all finite partitions $\pi = \{0 = t_0 < t_1 < \cdots t_n = t\}$ of $[0,t]$. Then

$$\sum_{k=0}^{n-1} |M_{(k+1)t/n} - M_{kt/n}|^2 \le \Delta_n \sum_{k=0}^{n-1} |M_{(k+1)t/n} - M_{kt/n}| \le \Delta_n V_t, \tag{15.1}$$

where $\Delta_n := \sup\{|M_u - M_v| : |u-v| \le 1/n, 0 \le u, v \le t\}$. By the continuity of $u \mapsto M_u$ on $[0,t]$, we have $\lim_n \Delta_n = 0$. It now follows from (15.1) and (11.1) that $\langle M\rangle_t = 0$, $\mathbf{P}$-a.s. Apply Lemma 14 to finish. ∎

Before turning to the statement of Itô's formula, we require a definition.

16. Itô process. An *Itô process* is a process of the form

$$X_t = X_0 + \int_0^t H_s\, dB_s + \int_0^t u_s\, ds, \qquad t \ge 0, \tag{16.1}$$

where $X_0 \in \mathcal{F}_0$, $H \in \mathcal{L}^2_{\rm loc}$ and u is progressively measurable and $\int_0^t |u_s|\, ds < \infty$ for all $t > 0$, $\mathbf{P}$-a.s. The decomposition (16.1) is unique: suppose the Itô process X admits a second such decomposition $X_t = X_0 + (K \bullet B)_t + \int_0^t v_s\, ds$. Subtracting we find that

$$\int_0^t (H_s - K_s)\, dB_s = \int_0^t (v_s - u_s)\, ds, \qquad t \ge 0. \tag{16.2}$$

The process on the left in (16.2) is a local martingale, while the one on the right has paths of bounded variation. Since both sides vanish at time 0, it follows from Corollary 15 that the two sides vanish for all t, almost surely. This proves that

$$\int_0^t v_s\, ds = \int_0^t u_s\, ds, \qquad \forall t \ge 0,$$

whence (upon differentiating in t)

$$v_t(\omega) = u_t(\omega), \qquad \text{for } \mathbf{P} \otimes \text{Leb-a.e. } (\omega, t) \in \Omega \times [0, \infty). \tag{16.3}$$

Moreover, because the local martingale on the left side of (16.2) vanishes, it has zero quadratic variation. That is,

$$\int_0^t (H_s - K_s)^2 \, ds = 0, \qquad \forall t \geq 0, \tag{16.4}$$

$\mathbf{P}$-a.s. Arguing as for (16.3), we deduce from (16.4) that

$$K_t(\omega) = H_t(\omega), \qquad \text{for } \mathbf{P} \otimes \text{Leb-a.e. } (\omega, t) \in \Omega \times [0, \infty). \tag{16.5}$$

This proves the uniqueness assertion.

Notice that if X is an Itô process, then its quadratic variation (in the sense of **11** is given by

$$\langle X \rangle_t = \int_0^t H_s^2 \, ds, \qquad \forall t \geq 0. \tag{16.6}$$

We now state, without proof, Itô's fundamental result. (Detailed proofs can be found in the books of Chung/Williams and Karatzas/Shreve listed on the course website.)

17. Theorem. [Itô's formula] *Let X be an Itô process as in (16.1) and let $f : \mathbf{R} \times [0, \infty) \to \mathbf{R}$ be of class $C^{2,1}$ (that is, twice continuously differentiable in its first argument and once continuously differentiable in its second). Then the composite process $t \mapsto f(X_t, t)$ is an Itô process with decomposition*

(17.1)

$$\begin{aligned} f(X_t, t) &= f(X_0, 0) + \int_0^t f_1'(X_s, s) \, dX_s + \int_0^t f_2'(X_s, s) \, ds + \frac{1}{2} \int_0^t f_{11}''(X_s, s) \, d\langle X \rangle_s \\ &= f(X_0, 0) + \int_0^t f_1'(X_s, s) H_s \, dB_s + \int_0^t [f_1'(X_s, s) u_s + f_2'(X_s, s) + \tfrac{1}{2} f_{11}''(X_s, s) H_s^2] \, ds. \end{aligned}$$

The first expression on the right side of (17.1) is a compact expression of Itô's formula; the second expression gives more explicitly the decomposition of $f(X)$ into stochastic integral and absolutely continuous terms.

The following special case of (1.71) is worth noting.

18. Corollary. *Let X be an Itô process as in (16.1) and let $f : \mathbf{R} \to \mathbf{R}$ be of class C^2. Then the composite process $f(X)$ is an Itô process with decomposition*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int_0^t f''(X_s) \, d\langle X \rangle_s \\ &= f(X_0) + \int_0^t f'(X_s) H_s \, dB_s + \int_0^t [f'(X_s) u_s + \tfrac{1}{2} f''(X_s) H_s^2] \, ds. \end{aligned} \tag{18.1}$$

19. Example. Suppose h is a $C^{2,1}$ function such that $\frac{1}{2}h''_{11} + h'_2$ vanishes identically. Then by (17.1) (with $H \equiv 1$ and $u \equiv 0$)

$$h(B_t, t) = h(B_0, 0) + \int_0^t h'_x(B_s, s)\, dB_s, \qquad t \geq 0, \tag{19.1}$$

which is a local martingale. If $h'_x(B_s, s) \in \mathcal{L}^2$, then the stochastic integral on the right side of (19.1) (hence also $h(B_t, t)$) is even a martingale. Examples of such functions are (i) $\exp(\lambda x - \lambda^2 t/2)$, (ii) $\sinh(\lambda x)e^{-\lambda^2 t/2}$, (iii) $\cosh(\lambda x)e^{-\lambda^2 t/2}$, (iv) $\sin(\lambda x)e^{\lambda^2 t/2}$, (v) $\cos(\lambda x)e^{\lambda^2 t/2}$.

20. Example. We know that B is a martingale. By Itô's formula,

$$B_t^2 = 2\int_0^t B_s\, dB_s + t.$$

Therefore, $H_t^{(2)} := B_t^2 - t$ is also a martingale. Next,

$$B_t^3 = 3\int_0^t B_s^2\, dB_s + 3\int_0^t B_s\, ds. \tag{20.1}$$

But,

$$\int_0^t B_s^2\, dB_s = \int_0^t H_s^{(2)}\, dB_s + \int_0^t s\, dB_s = \int_0^t H_s^{(2)}\, dB_s + tB_t - \int_0^t B_s\, ds, \tag{20.2}$$

the second equality resulting from the integration by parts formula (see **23** below). Feeding (20.2) into (20.1) we obtain the martingale

$$H_t^{(3)} := B_t^3 - 3tB_t = 3\int_0^t H_s^{(2)}\, dB_s.$$

This iteration procedure can be continued to produce a succession of polynomial functions of Brownian motion that are martingales, the so-called Hermite polynomials.

21. Example. The following example will be discussed in more detail in a separate handout. Fix $H \in \mathcal{L}^2_{\text{loc}}$ and define the local martingale M to be $H \bullet B$. Now take

$$Z_t := \exp(M_t - \tfrac{1}{2}\langle M\rangle_t), \qquad t \geq 0.$$

Using Itô's formula with $X_t = M_t - \frac{1}{2}\langle M\rangle_t$ and $f(x) = e^x$, one sees that

$$Z_t = 1 + \int_0^t Z_s\, dM_s.$$

In particular, Z is a (positive) local martingale. Under appropriate conditions on H, Z is even a martingale.

The following extension of **17** is often useful. We refer again to the books of Chung/Williams and Karatzas/Shreve for proofs.

22. Theorem. [Multivariate Itô Formula] *Let $X^1, \dots, X^n$ be Itô processes:*

$$X_t^k = X_0^k + \int_0^t H_s^{(k)}\, dB_s + \int_0^t u_s^{(k)}\, ds, \qquad k = 1, 2, \dots, n,$$

and let $f : \mathbf{R}^n \to \mathbf{R}$ be of class C^2. Then writing $X_t = (X_t^1, \dots, X_t^n)$,

$$f(X_t) = f(X_0) + \sum_{k=1}^{n} \int_0^t \frac{\partial f}{\partial x_k}(X_s) dX_s^k + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} f(X_s)\, d\langle X^j, X^k \rangle_s, \tag{22.1}$$

where

$$\langle X^j, X^k \rangle_t = \int_0^t H_s^{(j)} H_s^{(k)}\, ds. \tag{22.2}$$

23. Example. Here is the product rule for itô processes. Let $X_t = X_0 + \int_0^t H_s\, dB_s + \int_0^t u_s\, ds$ and $Y_t = Y_0 + \int_0^t K_s\, dB_s + \int_0^t v_s\, ds$ be Itô processes. Let us take $X^1 = X$, $X_2 = Y$ and $f(x, y) = xy$ in (22.1) to obtain

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \int_0^t H_s K_s\, ds.$$