Math 280C, Spring 2005

Strong Markov Property

Notation is that introduced in class.

Theorem. (Strong Markov Property of Brownian Motion). *If T is an $(\mathcal{F}_{t+})$-stopping time and μ is any initial distribution, then*

$$\mathbf{E}_\mu[F\circ\theta_T|\mathcal{F}_{T+}] = \mathbf{E}_{B(T)}(F), \qquad \mathbf{P}_\mu\text{-a.s. } on\ \{T<\infty\} \tag{1}$$

for each $F \in b\mathcal{F}$. [Here $B(T)$ is alternate notation for B_T.]

Proof. The right side of (1) is the composition of the $\mathcal{F}_{T+}$-measurable map $\omega \mapsto B_{T(\omega)}(\omega)$ with the Borel measurable map $x \mapsto \mathbf{E}_x[F]$, and is therefore $\mathcal{F}_{T+}$-measurable. It remains to show that

$$\mathbf{E}_\mu[H\cdot F\circ\theta_T] = \mathbf{E}_\mu\left[H\cdot \mathbf{E}_{B(T)}[F]\right] \tag{2}$$

for every $H \in b\mathcal{F}_{T+}$. In view of the monotone class theorem for functions, it suffices to prove (2) for F of the special form

$$F(\omega) = \prod_{k=1}^n f_k(B_{t_k}(\omega))$$

where n is a positive integer, $0 < t_1 < t_2 < \cdots < t_n$, and each $f_k : \mathbf{R} \to \mathbf{R}$ is a bounded continuous function. It has been proved in class that for such an F, the function $x \mapsto \mathbf{E}_x[F]$ is (bounded and) continuous. We proceed by approximating the stopping time T as follows:

$$T_n(\omega) = \begin{cases} (k+1)2^{-n}, & \text{if } k2^{-n} \le T(\omega) < (k+1)2^{-n},\ k = 0,1,2,\ldots; \\ +\infty, & \text{if } T(\omega) = +\infty. \end{cases}$$

Then, for $t > 0$,

$$\{T_n \le t\} = \cup_{k:(k+1)2^{-n}\le t}\{k2^{-n} \le T < (k+1)2^{-n}\} \in \mathcal{F}_t,$$

since $\{T < (k+1)2^{-n}\} \in \mathcal{F}_{(k+1)2^{-n}+}$ because T is an $(\mathcal{F}_{t+})$-stopping time. Thus T_n is a stopping time, and it is clear that $T_n(\omega)$ decreases to $T(\omega)$ for each ω. Consequently, because $t \mapsto B_t(\omega)$ is continuous, $\lim_{n\to\infty} B_{T_n}(\omega) = B_T(\omega)$, and so

$$\lim_{n\to\infty} \mathbf{E}_{B_{T_n(\omega)}(\omega)}[F] = \mathbf{E}_{B_{T(\omega)}(\omega)}[F].$$

Therefore, noting that $\{T<\infty\}=\{T_n<\infty\}$,

$$\begin{aligned} \mathbf{E}_\mu\left[H\cdot \mathbf{E}_{B(T)}[F];T<\infty\right] &= \mathbf{E}_\mu\left[H\cdot \lim_n \mathbf{E}_{B(T_n)}[F];T_n<\infty\right] \\ &= \lim_n \mathbf{E}_\mu\left[H\cdot \mathbf{E}_{B(T_n)}[F];T_n<\infty\right] \end{aligned} \tag{3}$$

But, by the law of total probability and Fubini's theorem,

$$\begin{aligned} \mathbf{E}_\mu\left[H\cdot \mathbf{E}_{B(T_n)}[F];T_n<\infty\right] &= \sum_{k=0}^\infty \mathbf{E}_\mu\left[H\cdot \mathbf{E}_{B((k+1)2^{-n})}[F];T_n=(k+1)2^{-n}\right] \\ &= \sum_{k=0}^\infty \mathbf{E}_\mu\left[H\cdot F\circ\theta_{(k+1)2^{-n}};T_n=(k+1)2^{-n}\right] \\ &= \mathbf{E}_\mu\left[H\cdot F\circ\theta_{T_n};T_n<\infty\right]. \end{aligned} \tag{4}$$

The second equality above results from the simple Markov property because (i)

$$\{T_n=(k+1)2^{-n}\}=\{k2^{-n}\le T<(k+1)2^{-n}\}\in\mathcal{F}_{(k+1)2^{-n}},$$

and (ii) $H\cdot 1_{\{k2^{-n}\le T<(k+1)2^{-n}\}}\in\mathcal{F}_{(k+1)2^{-n}}$ as H is $\mathcal{F}_{T+}$-measurable. On the other hand, because of the special choice of F, the map $t\mapsto F(\theta_t\omega)$ is continuous, so $\lim_n F(\theta_{T_n(\omega)}\omega)=F(\theta_{T(\omega)}\omega)$, at least for ω in $\{T<\infty\}$. It now follows from the dominated convergence theorem that

$$\lim_n \mathbf{E}_\mu[H\cdot F\circ\theta_{T_n};T_n<\infty]=\mathbf{E}_\mu[H\cdot F\circ\theta_T;T<\infty]. \tag{5}$$

Combining (3), (4), and (5), we obtain (2). ∎

The following reformulation of the strong Markov property for Brownian motion is often convenient. For simplicity I assume that T is finite.

Corollary. *If T is an $(\mathcal{F}_{t+})$-stopping time and μ is any initial distribution such that $\mathbf{P}_\mu[T<\infty]=1$, then the process $B^{(T)}$ defined by*

$$B_t^{(T)}:=B_{T+t}-B_T,\quad t\ge 0,$$

has the same distribution as Brownian motion started at 0, and is independent of $\mathcal{F}_{T+}$.

Proof. For this proof only let $\tau_x:\Omega\to\Omega$ be defined by $(\tau_x\omega)(t):=\omega(t)-x$. Then $B^{(T)}(\omega)$ is equal to $\theta_{T(\omega)}=\tau_x[\theta_{T(\omega)}\omega]$ when x is set equal to $B_T(\omega)$. Consequently, if $H\in b\mathcal{F}_{T+}$ and $F\in b\mathcal{F}$,

$$\begin{aligned} \mathbf{E}_\mu[H\cdot F(B^{(T)})] &= \mathbf{E}_\mu[H\cdot F(\tau_{B(T)}\theta_T)] \\ &= \mathbf{E}_\mu[H\cdot \mathbf{E}_\mu[F(\tau_{B(T)}\theta_T)|\mathcal{F}_{T+}]=\mathbf{E}_\mu[H\cdot f(B_T)], \end{aligned}$$

where $f(x) = \mathbf{E}_x[F(\tau_x)]$. But Brownian motion started at x is equal in distribution to Brownian motion started at 0 and then translated by x. This translation is cancelled by τ_x, so $f(x) = \mathbf{E}_x[F(\tau_x)] = \mathbf{E}_0[F]$. Thus,

$$\mathbf{E}_\mu[H \cdot F(B^{(T)})] = \mathbf{E}_\mu\left[H \cdot \mathbf{E}_0[F]\right] = \mathbf{E}_\mu[H] \cdot \mathbf{E}_0[F],$$

which proves both assertions of the theorem. ∎

Example. The Brownian transition density $p_t(x,y)$ induces an operator P_t on functions as follows: If $f : \mathbf{R} \to \mathbf{R}$ is Borel measurable and bounded (or positive) then we define $P_t f$ as the function

$$P_t f(x) := \int_{\mathbf{R}} p_t(x,y) f(y)\, dy, \qquad x \in \mathbf{R}.$$

Evidently, $\|P_t f\|_\infty \le \|f\|_\infty$, and (Scheffe's lemma again!) $x \mapsto P_t f(x)$ is a continuous function if f is bounded.

Let T be a stopping time, and let us compute the conditional expectation $\mathbf{E}_\mu[f(B_t)|\mathcal{F}_{T+}]$, where $t > 0$ is fixed. Observe that if $T(\omega) < t$ then

$$f(B_t(\omega)) = g(\theta_{T(\omega)}, T(\omega)),$$

where $g(\omega', s) := f(w'(t-s))$. Since T is $\mathcal{F}_{T+}$-measurable, we can use (1) to compute

$$\mathbf{E}_\mu[f(B_t)|\mathcal{F}_{T+}] = 1_{\{T \ge t\}} f(B_t) + 1_{\{T<t\}} h(B_T, T),$$

where, for $0 \le s < t$,

$$h(x,s) := \mathbf{E}_x[g(\cdot, s)] = \mathbf{E}_x[f(B_{t-s})] = P_{t-s} f(x).$$

Thus,

$$\mathbf{E}_\mu[f(B_t)|\mathcal{F}_{T+}] = 1_{\{T \ge t\}} f(B_t) + 1_{\{T<t\}} P_{t-T} f(B_T).$$