Math 285A, Spring 2006

Homework 6, due at noon on June 16, 2006

1. Exercise 9.1, page 173 of the text.

2. Exercise 9.2, page 173–174 of the text.

3. Let X_t, $t \geq 0$, be standard one-dimensional Brownian motion, with $X_0 = 0$. Define, for $t \geq 0$ and $k = 0, 1, 2, \ldots$,

$$m_k(t) = \mathbf{E}[X_t^{2k}].$$

Use Itô's formula (and then take expectations—the martingale terms have mean zero!) to show that

$$m_k(t) = t(2k-1)m_{k-1}(t), \qquad t > 0, k = 1, 2, 3, \ldots.$$

Deduce by recursion that

$$m_k(t) = 1 \cdot 3 \cdot 5 \cdots (2k-1)t^k, \qquad t > 0, k = 1, 2, \ldots.$$

4. Let f and g be (non-random) continuous functions on $[0,1]$ and suppose there are constants C and D such that

$$C + \int_0^1 f(s)\,dX_s = D + \int_0^1 g(s)\,dX_s.$$

Show that $C = D$ and that $f(s) = g(s)$ for all $s \in [0,1]$.
[Hints: The martingale $M_t = \int_0^t [f(s) - g(s)]\,dX_s$ satisfies $M_0 = 0$ and $M_1 = D - C$. Deduce from this that $C = D$. Consequently, $M_1 = 0$. Now use the formula for the variance of a stochastic integral to see that $\int_0^1 [f(s) - g(s)]^2\,ds = 0$.]