In what follows, \((\Omega, \mathcal{F}, P)\) is the canonical sample space of the Brownian motion \((B_t)_{t \geq 0}\) with \(B_0 = 0\); other notation is that used in class.

Given \(H \in L^2_{\text{loc}}\) let \(M\) denote the associated local martingale:

\[
M_t := \int_0^t H_s \, dB_s, \quad t \geq 0.
\]

Now define a strictly positive continuous adapted process \(Z\) by

\[
Z_t := \exp \left( M_t - \frac{1}{2} \langle M \rangle_t \right), \quad t \geq 0.
\]

Clearly \(Z_0 = 1\), and it follows easily from Itô’s formula that

\[
Z_t = 1 + \int_0^t Z_s \, dM_s = 1 + \int_0^t Z_s H_s \, dB_s.
\]

In other words, \(Z\) solves the “stochastic differential equation” (SDE)

\[
dZ_t = Z_t \, dM_t, \quad t \geq 0,
\]

with initial condition

\[
Z_0 = 1.
\]

For this reason we refer to \(Z\) as the stochastic exponential of \(M\).

In view of (3), \(Z\) is a local martingale. Let \((T_n)\) reduce \(Z\). Then for each \(n\)

\[
1 = \mathbb{E}[Z_0] = \mathbb{E}[Z_{t \wedge T_n}],
\]

Because \(Z\) is a positive local martingale we can appeal to Fatou’s lemma to deduce that

\[
\mathbb{E}[Z_t] = \mathbb{E}[\lim_n Z_{t \wedge T_n}] \leq \liminf_n \mathbb{E}[Z_{t \wedge T_n}] = 1.
\]

Thus \(Z_t\) is integrable for each \(t \geq 0\). A second application of Fatou’s lemma shows that \(Z\) is a supermartingale. In particular, \(Z\) is a martingale if and only if \(\mathbb{E}[Z_t] = 1\) for all \(t > 0\).
Theorem 1. $Z$ is the unique solution of the initial value problem (4), (5).

Proof. Let $Y$ be a second (continuous) local martingale such that $Y_t = 1 + \int_0^t Y_s \, dM_s$ for all $t \geq 0$. Because $Z_t > 0$ for all $t$ we can apply Itô’s formula to the ratio $Y/Z$:

$$d(Y_t Z_t^{-1}) = Y_t \, d(Z_t^{-1}) + Z_t^{-1} \, dY_t + d(Y, Z^{-1})_t$$

$$= -Y_t Z_t^{-2} \, dZ_t + Y_t Z_t^{-3} \, d(Z)_t + Z_t^{-1} \, dY_t - Z_t^{-2} \, d(Y, Z)_t$$

$$= Y_t Z_t^{-1} \, dM_t + Y_t Z_t^{-1} \, d(M)_t + Z_t^{-1} Y_t \, dM_t - YZ_t^{-1} \, d(M)_t$$

$$= 0$$

Thus, $Y_t/Z_t = Y_0/Z_0 = 1$ for all $t > 0$, so $Y$ and $Z$ are identical. \(\square\)

The process $Z$ is most useful when it is a martingale. We shall develop a simple sufficient condition under which this is true. As preparation we require the following lemma, which is of independent interest.

**Gronwall’s Lemma.** Let $g$ and $b$ be non-negative Borel measurable functions defined on $[0, \infty)$ and let $a$ be a non-negative constant. If, for some $t_0 > 0$, we have $\int_0^{t_0} b(s) \, ds < \infty$ and

$$g(t) \leq a + \int_0^t g(s) b(s) \, ds, \quad \forall t \in [0, t_0],$$

then

$$g(t) \leq a \exp\left(\int_0^t b(s) \, ds\right), \quad \forall t \in [0, t_0].$$

Proof. Define $B(t) := \int_0^t b(s) \, ds$ and $G(t) := \int_0^t g(s) b(s) \, ds$ for $t \in [0, t_0]$. Then

$$\frac{d}{dt} \left[ e^{-B(t)} G(t) \right] = e^{-B(t)} b(t) \left[ g(t) - G(t) \right] \leq a e^{-B(t)} b(t),$$

for a.e. $t \in [0, t_0]$. Integrating the extreme terms in (10) we find that

$$e^{-B(t)} G(t) \leq \int_0^t a e^{-B(s)} b(s) \, ds = a \left(1 - e^{-B(t)}\right), \quad \forall t \in [0, t_0].$$

Thus, $G(t) \leq a (e^{B(t)} - 1)$, so

$$g(t) \leq a + G(t) \leq a e^{B(t)}, \quad t \in [0, t_0],$$

as claimed. \(\square\)
**Theorem 2.** Suppose that $H \in \mathcal{L}^2$ satisfies the bound $|H_s(\omega)| \leq f(s)$ for all $(\omega, s) \in \Omega \times [0, \infty)$, where $\int_0^t |f(s)|^2 \, ds < \infty$ for each $t > 0$. If $Z$ is the stochastic exponential associated with $H$ as in (1) and (2), then $Z$ is a square-integrable martingale.

**Proof.** From Itô’s formula,

$$Z_t^2 = 1 + 2 \int_0^t Z_s \, dZ_s + \int_0^t Z_s^2 H_s^2 \, ds. \quad (13)$$

In particular, $Z_t^2 - \int_0^t Z_s^2 H_s^2 \, ds = Z_t^2 - \langle Z \rangle_t$ is a local martingale. Let $(T_n^1)$ be a sequence of stopping times reducing this local martingale. Let $(T_n^2)$ be a sequence of stopping times reducing the local martingale $Z$. Then $T_n := T_n^1 \wedge T_n^2$ defines a sequence of stopping times that reduces both $Z$ and $Z^2 - \langle Z \rangle$. In particular,

$$E[Z_{t \wedge T_n}^2] = 1 + E \left[ \int_0^{t \wedge T_n} Z_s^2 H_s^2 \, ds \right] \quad (14)$$

Let us fix $n$ for a moment and define $g(t) := E[Z_{t \wedge T_n}^2]$ for $t \in [0, \infty)$. Then (14) implies

$$g(t) \leq 1 + \int_0^t g(s) |f(s)|^2 \, ds, \quad t \in [0, \infty). \quad (15)$$

Feeding (15) into Gronwall’s lemma we deduce that

$$E[Z_{t \wedge T_n}^2] \leq \exp \left( \int_0^t |f(s)|^2 \, ds \right), \quad t \in [0, \infty), n = 1, 2, \ldots. \quad (16)$$

Now Doob’s inequality applied to the stopped process $Z_{t \wedge T_n}$ (a u.i. martingale!) yields

$$E \left[ \sup_{0 \leq s \leq t \wedge T_n} Z_s^2 \right] \leq 4E[Z_{t \wedge T_n}^2] \leq 4 \exp \left( \int_0^t |f(s)|^2 \, ds \right), \quad t \in [0, \infty). \quad (17)$$

It follows from (17) and the “crystal ball” condition that for each $t > 0$ the collection of random variables $\{Z_{s \wedge T_n} : s \in [0, t], n \in \mathbb{N} \}$ is uniformly integrable. In particular, $Z_{t \wedge T_n}$ converges both a.s. and (more importantly) in $L^1$ to $Z_t$ as $n \to \infty$. Since $(Z_{t \wedge T_n})_{0 \leq t \leq t_0}$ is a martingale, so is its $L^1$ limit $(Z_t)_{0 \leq t \leq t_0}$. That this martingale is square integrable follows immediately from (16) and Fatou’s lemma. \[\square\]
Example 1. (Cf. Problem 2, Homework 6): If \( f \) is a measurable function on \([0, \infty)\) with 
\[ \int_0^t [f(s)]^2 \, ds < \infty \]
for each \( t > 0 \), then

\[
Z_t := \exp \left( \int_0^t f(s) \, dB_s - \frac{1}{2} \int_0^t [f(s)]^2 \, ds \right), \quad t \geq 0,
\]

is a strictly positive martingale. From this one can deduce, as in the homework problem just cited, that \( M_t := \int_0^t f(s) \, dB_s \) is normally distributed with mean 0 and variance \( \int_0^t [f(s)]^2 \, ds \).

Sharper criteria for \( Z \) to be a true martingale are known, but their proofs are more delicate. Let us state the two most well known, without proofs. Notation is as in (1) and (2).

**Theorem 3.** [Novikov] If

\[
E \left[ \exp \left( \frac{1}{2} \langle M \rangle_t \right) \right] < \infty,
\]

then \( E[Z_t] = 1 \), in which case \((Z_s)_{0 \leq s \leq t}\) is a martingale.

**Theorem 4.** [Kazamaki] If

\[
\sup_{0 \leq s \leq t} E \left[ \exp \left( \frac{1}{2} M_s \right) \right] < \infty,
\]

then \( E[Z_t] = 1 \), in which case \((Z_s)_{0 \leq s \leq t}\) is a martingale.

**Remark 1.** The form (2) for the local martingale \( Z \) may seem very special, but in fact any \emph{strictly positive} local martingale has this form. This is discussed in some detail in the handout on Girsanov’s theorem.