Math 285, Spring 2009

Homework 10, not to be handed in

1. Exercise 9.5, page 229 of the text.

2. Exercise 9.6, page 230 of the text.

3. Let W_t, $t \geq 0$, be standard one-dimensional Brownian motion, with $W_0 = 0$. Define, for $t \geq 0$ and $k = 0, 1, 2, \ldots$,

$$m_k(t) = \mathbf{E}[W_t^{2k}].$$

Use Itô's formula (and then take expectations—the martingale terms have mean zero!) to show that

$$m_k(t) = t(2k-1)m_{k-1}(t), \qquad t > 0, k = 1, 2, 3, \ldots.$$

Deduce by recursion that

$$m_k(t) = 1 \cdot 3 \cdot 5 \cdots (2k-1)t^k, \qquad t > 0, k = 1, 2, \ldots.$$

4. Let W_t, $t \geq 0$, be standard one-dimensional Brownian motion, with $W_0 = 0$. Let f and g be (non-random) continuous functions on $[0, 1]$ and suppose there are constants C and D such that

$$C + \int_0^1 f(s)\, dW_s = D + \int_0^1 g(s)\, dW_s.$$

Show that $C = D$ and that $f(s) = g(s)$ for all $s \in [0, 1]$.
[Hints: The martingale $M_t = \int_0^t [f(s) - g(s)]\, dW_s$ satisfies $M_0 = 0$ and $M_1 = D - C$. Deduce from this that $C = D$. Consequently, $M_1 = 0$. Now use the formula for the variance of a stochastic integral to see that $\int_0^1 [f(s) - g(s)]^2\, ds = 0$.]