Math 285, Spring 2009

An Integral

The following integral arises in the calculation of the Laplace transform of the density function of the first passage time T_b.

Proposition. *For $\alpha > 0$ and $\beta \geq 0$,*

$$\int_0^\infty e^{-\alpha t - \beta/t} t^{-1/2}\, dt = \sqrt{\frac{\pi}{\alpha}} e^{-2\sqrt{\alpha\beta}}. \tag{1}$$

Proof. Fix $\alpha > 0$ and view the left side of (1) as a function of β; call this function φ. Observe that

$$\varphi(0) = \int_0^\infty e^{-\alpha t} t^{-1/2}\, dt = \alpha^{-1/2} \int_0^\infty e^{-u} u^{-1/2}\, du = \alpha^{-1/2}\Gamma(1/2) = \sqrt{\pi/\alpha}, \tag{2}$$

where the second equality results from the change of variables $u := \alpha t$. Also, differentiating under the integral sign (justification left to the reader),

$$\varphi'(\beta) = -\int_0^\infty e^{-\alpha t - \beta/t} t^{-3/2}\, dt. \tag{3}$$

Let us make the change of variables $s := \beta/(\alpha t)$ on the right side of (3). Then $dt = -(\beta/\alpha)\, ds/s^2$, so

$$\varphi'(\beta) = -\sqrt{\frac{\alpha}{\beta}} \int_0^\infty e^{-\beta/s - \alpha s} s^{-1/2}\, ds = -\sqrt{\frac{\alpha}{\beta}} \varphi(\beta). \tag{4}$$

The equality between the extreme terms in (4) is a simple first order linear differential equation; separating variables and integrating we find that

$$\varphi(\beta) = C \cdot e^{-2\sqrt{\alpha\beta}}.$$

The constant C is the initial value $\varphi(0) = \sqrt{\pi/\alpha}$ exhibited in (2). ∎

Corollary 1. *For $\alpha > 0$ and $\beta \geq 0$,*

$$\int_0^\infty e^{-\alpha t - \beta/t} t^{-3/2}\, dt = \sqrt{\pi/\beta} e^{-2\sqrt{\alpha\beta}}. \tag{5}$$

Proof. Differentiate with respect to β in (1). ∎

Corollary 2. *For $\alpha > 0$ and $b > 0$,*

$$\int_0^\infty e^{-\alpha t} \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t}\, dt = \exp^{-b\sqrt{2\alpha}}.$$

Proof. This follows immediately from (5) with $\beta = b^2/2$. ∎