Pr. 7.3.4.  
\[ E[L] = \int_{0}^{1} \int_{0}^{x} x \, dy \, dx + \int_{0}^{1} \int_{x}^{1} (1 - x) \, dy \, dx \]
\[ = \int_{0}^{1} x^2 \, dx + \int_{0}^{1} (1 - x)^2 \, dx \]
\[ = 2 \int_{0}^{1} x^2 \, dx = 2/3 > 1/2. \]

Ex. 7.4.2. If \( f(x) = 2x \) for \( 0 < x < 1 \), then \( F(x) = x^2 \) for \( 0 \leq x < 1 \) and \( F(x) = 1 \) for \( x \geq 1 \). Notice that \( \mu = 2/3 \) and \( \sigma^2 = 1/18 \) in the present case. The limit density of the excess life is
\[ \frac{1 - F(x)}{\mu} = \begin{cases} \frac{3}{2} (1 - x^2), & \text{if } 0 < x < 1, \\ 0, & \text{if } x \geq 1. \end{cases} \]
The limiting mean excess life is therefore
\[ \int_{0}^{1} x \frac{3}{2} (1 - x^2) \, dx = \frac{3}{2} [1/2 - 1/4] = 3/8. \]
This is consistent with the formula on page 444 of the text, which indicates that the desired mean is
\[ \frac{\sigma^2 + \mu^2}{2\mu} = \frac{1/18 + 4/9}{4/3} = \frac{1/2}{4/3} = 3/8. \]

Ex. 7.4.3. For this density we have
\[ \mu(T) = \int_{0}^{T} (1 - x^2) \, dx = T - T^3/3 \]
for \( 0 < T < 1 \). Therefore
\[ C(T) = \frac{3 + 12T^2}{T(3 - T^2)}. \]
The critical point equation \( C'(T) = 0 \) is equivalent to
\[ 4T^4 + 15T^2 - 3 = 0. \]
The quadratic formula yields
\[ T^2 = \frac{-15 \pm \sqrt{273}}{8} = 0.190338955, -3.940338955. \]
The relevant root is the (real) positive one, so $T = 0.436278529$.

**Ex. 7.4.5.** The excess life density is

$$2(1-x)$$

for $0 < x < 1$. (The backwards triangle density.)

**Pr. 7.4.1.** For this renewal function,

$$\lim_{t \to \infty} \frac{M(t)}{t} = 1,$$

while the renewal theorem tells us that this limit equals $1/\mu$. Therefore $\mu = 1$. Similarly, for this renewal function,

$$\lim_{t \to \infty} [M(t) - t] = 1,$$

while the renewal theorem tells us that this limit equals $\frac{\sigma^2 - \mu^2}{2\mu^2}$. Therefore

$$1 = \frac{\sigma^2 - 1}{2},$$

so $\sigma^2 = 3$.

**Pr. 7.4.5.** (a) Let $S_0$ be a typical sojourn in state 0. Then

$$P[S_0 \geq n] = P[X_1 = X_2 = \cdots = X_{n-1} = 0 | X_0 = 0] = (.3)^{n-1}, \quad n = 1, 2, \ldots.$$ 

Consequently,

$$E[S_0] = \sum_{n=1}^{\infty} P[S_0 \geq n] = \sum_{n=1}^{\infty} (.3)^{n-1} = 10/7.$$ 

In the same way, if $S_2$ be a typical sojourn in state 2 then

$$E[S_2] = 2.$$ 

(b) In view of the renewal theorem, we need to compute the mean length of an excursion away from state 1. The reciprocal of this mean will be the desire fraction. Starting in state 1 the chain moves to state 0 (with probability .6) or to state 2 (with probability .4). It then sojourns in 0 or 2 (as the case may be) and ends the cycle by returning to 1. The mean length of the excursion is

$$1 + .6 \cdot (10/7) + .4 \cdot 2 = 93/35 = 2.66.$$ 

Therefore the long-run fraction of time spent in state 1 is $35/93 = 0.376$. As a check on this, notice that the stationary distribution for this chain is

$$\pi = [30/93, 35/93, 38/93].$$