Ex. 6.1.1. \( P_0(t) = e^{-t}. \) For the rest I use the recursion (formula (6.5) on page 280 of the text)

\[
P_n(t) = \lambda_{n-1} \int_0^t e^{-\lambda_n(t-s)} P_{n-1}(s) \, ds,
\]

for \( n = 1, 2, 3: \)

\[
P_1(t) = \int_0^t e^{-3(t-s)} P_0(s) \, ds = e^{-3t} \int_0^t e^{3s} e^{-s} \, ds
\]

\[
= \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}.
\]

\[
P_2(t) = 3 \int_0^t e^{-2(t-s)} P_1(s) \, ds = \frac{3}{2} \int_0^t e^{-2(t-s)} (e^{-s} - e^{-3s}) \, ds
\]

\[
= \frac{3}{2} e^{-2t} \int_0^t (e^s - e^{-s}) \, ds = \frac{3}{2} e^{-2t} \left[ (e^t - 1) - (1 - e^{-t}) \right]
\]

\[
= \frac{3}{2} (e^{-t} - 2e^{-2t} + e^{-3t}).
\]

\[
P_3(t) = 2 \int_0^t e^{-5(t-s)} P_2(s) \, ds = 3 e^{-5t} \int_0^t e^{5s} (e^{-s} - 2e^{-2s} + e^{-3s}) \, ds
\]

\[
= 3 e^{-5t} \int_0^t (e^{4s} - 2e^{3s} + e^{2s}) \, ds
\]

\[
= 3 e^{-5t} \left[ \frac{1}{4} (e^{4t} - 1) - \frac{2}{3} (e^{3t} - 1) + \frac{1}{2} (e^{2t} - 1) \right]
\]

\[
= \frac{3}{4} e^{-t} - 2e^{-2t} + \frac{3}{2} e^{-3t} - \frac{1}{4} e^{-5t}.
\]

Ex. 6.1.2. (a) \( W_3 = S_0 + S_1 + S_2, \) so

\[
E[W_3] = E[S_0] + E[S_1] + E[S_2]
\]

\[
= \lambda_0^{-1} + \lambda_1^{-1} + \lambda_2^{-1}
\]

\[
= 1 + \frac{1}{3} + \frac{1}{2} = \frac{11}{6}.
\]

(b) Similarly, \( E[W_1] = 1 \) and \( E[W_2] = 4/3, \) so \( E[W_1 + W_2 + W_3] = 1 + 4/3 + 11/6 = 25/6. \)

(c) The variance of \( W_3 \) is the sum of the variances of \( S_0, S_1, \) and \( S_2. \) We know that the variance of an exponentially distributed random variable with parameter \( \lambda \) is \( 1/\lambda^2. \) Therefore,

\[
\text{Var}[W_3] = 1 + \frac{1}{9} + \frac{1}{4} = \frac{49}{36}.
\]

Ex. 6.1.5. According to formula (6.10) (page 282 of the text), if \( X(0) = 1 \) then \( X(t) \) has the geometric distribution with parameter \( p = e^{-\beta t}. \) From known formulas for the mean and variance of a geometric random variable we deduce that

\[
E[X(t)] = e^{\beta t}
\]
and
\[ \text{Var}[X(t)] = e^{2\beta t}(1 - e^{-\beta t}). \]

(Cf. the formulae for the moments of $Z'$ on page 21 of the text.)

**Pr. 6.1.3.** If, at time $t$, there are $X(t)$ infected individuals, then at that time there are $N - X(t)$ susceptible individuals in the population. Since the individual infection rate (for each infected/susceptible pair) is $\alpha$, the total infection rate, when $X(t) = k$, is $\alpha k(N - k)$. That is, $\lambda_k = \alpha k(N - k)$ for $k = 0, 1, 2, \ldots, N$.

**Pr. 6.1.8.** Evidently,
\[ P_0(t + h) = P_0(t)(1 - \beta h) + P_1(t)\alpha h + o(h), \quad h \to 0. \]
Consequently,
\[ \frac{P_0(t + h) - P_0(t)}{h} = -\beta P_0(t) + \alpha P_1(t) + \frac{o(h)}{h}, \]
whence
\[ P_0'(t) = -\beta P_0(t) + \alpha P_1(t). \]
Similarly,
\[ P_1'(t) = -\alpha P_1(t) + \beta P_0(t). \]
Because $P_1(t) = 1 - P_0(t)$, equation (6.1.8.1) can be rewritten as
\[ P_0'(t) = -\beta P_0(t) + \alpha. \]

Thus,
\[ P_0'(t) + (\beta + \alpha)P_0(t) = \alpha, \]
so
\[ \frac{d}{dt} \left( e^{(\alpha + \beta)t} P_0(t) \right) = \alpha e^{(\alpha + \beta)t}. \]
Integrating we find that
\[ e^{(\alpha + \beta)t} P_0(t) - 1 = \frac{\alpha}{\alpha + \beta}(e^{(\alpha + \beta)t} - 1), \]
because $P_0(0) = 1$. It follows that
\[ P_0(t) = \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} e^{-(\alpha + \beta)t}. \]
Because $P_1(t) = 1 - P_0(t)$, we also have
\[ P_1(t) = -\frac{\beta}{\alpha + \beta} e^{-(\alpha + \beta)t}. \]

**Pr. 6.1.9.** Let $P_k(t) = \mathbb{P}[N(t) = k]$ for $k = 0, 1, 2, \ldots$ and $M(t) = \mathbb{E}[N(t)]$. Let us also write $\eta(t)$ for $\mathbb{P}[N(t) \text{ is even}]$. (This was called $P_0(t)$ in Problem 6.1.8.) We know that
\[ P_{2k}'(t) = -\beta P_{2k}(t) + \alpha P_{2k-1}(t), \quad k = 1, 2, \ldots, \]
and
\[ P'_{2k+1} = -\alpha P_{2k+1}(t) + \beta P_{2k}(t), \quad k = 0, 1, 2, \ldots. \]
Therefore,

\[ M'(t) = \sum_{k=1}^{\infty} 2kP'_{2k}(t) + \sum_{k=0}^{\infty} (2k+1)P'_{2k+1}(t) \]

\[ = -\beta \sum_{k=1}^{\infty} 2kP_{2k}(t) + \alpha \sum_{k=1}^{\infty} 2kP_{2k-1}(t) - \alpha \sum_{k=0}^{\infty} (2k+1)P_{2k+1}(t) + \beta \sum_{k=0}^{\infty} (2k+1)P_{2k}(t) \]

\[ = \beta \eta(t) + \alpha (1 - \eta(t)) \]

\[ = \alpha + (\beta - \alpha) \eta(t). \]

Using the formula for \( \eta(t) \) from problem 6.1.8, we see that

\[ M'(t) = \alpha + (\beta - \alpha) \left( \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} e^{-(\alpha + \beta)t} \right). \]

Since \( M(0) = 0 \), we must have

\[ M(t) = \frac{2\alpha \beta}{\alpha + \beta} t + \frac{\beta(\beta - \alpha)}{(\alpha + \beta)^2} (1 - e^{-(\alpha + \beta)t}). \]

Ex. 6.2.1. Using the formulas on page 287 of the text (and a little patience), we find that

\[ P_3(t) = e^{-5t} \]
\[ P_2(t) = \frac{5}{3} [e^{-2t} - e^{-5t}] \]
\[ P_1(t) = \frac{5}{3} [2e^{-2t} - 3e^{-3t} + e^{-5t}], \]

and of course \( P_0(t) = 1 - P_1(t) - P_2(t) - P_3(t) \), so that

\[ P_5(t) = 1 - 5e^{-2t} + 5e^{-3t} - e^{-5t}. \]

As a check, note that \( P_3(0) = 1, P_2(0) = P_1(0) = P_0(0) = 0 \), and

\[ P'_3(t) = -5P_3(t) \]
\[ P'_2(t) = -2P_2(t) + 5P_3(t) \]
\[ P'_1(t) = -3P_1(t) + 2P_2(t), \]

as expected.

Alternatively, you can use the formula \( P_3(t) = e^{-\mu_3t} \), and the recursion discussed in class:

\[ P_k(t) = \mu_{k+1} \int_0^t e^{-\mu_k(t-s)} P_{k+1}(s) \, ds, \quad k = 2, 1, 0, \]

to compute (in succession) \( P_2(t) \), \( P_1(t) \), and \( P_0(t) \).

Ex. 6.2.2. (a) \( W_3 = S_3 + S_2 + S_1 \), so

\[ \mathbb{E}[W_3] = \mathbb{E}[S_3] + \mathbb{E}[S_2] + \mathbb{E}[S_1] = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} = \frac{31}{30}. \]

(b) Similarly, \( \mathbb{E}[W_2] = 5/6 \) and \( \mathbb{E}[W_1] = 1/3 \), so \( \mathbb{E}[W_1 + W_2 + W_3] = 11/5. \)
(c) $\text{Var}[W_3] = \text{Var}[S_3] + \text{Var}[S_2] + \text{Var}[S_1] = 1/25 + 1/4 + 1/9 = 361/900 = .4011\ldots$

**Pr. 6.2.2.** Since the death rates are all the same (namely $\theta$), the sojourn times in the various states all have the same exponential distribution, and their sums have gamma distributions. More precisely, for $k = 1, 2, \ldots N$, the random variable $W_k = S_N + S_{N-1} + \cdots + S_{N-k+1}$ has the gamma distribution with parameters $\theta$ and $k$; that is, the density function of $W_k$ is

$$f_{W_k}(t) = \frac{\theta^k t^{k-1} e^{-\theta t}}{(k-1)!}, \quad t > 0.$$ 

Therefore

$$\mathbb{P}[X(t) = n] = \mathbb{P}[W_{N-n} \leq t < W_{N-n+1}] = \mathbb{P}[W_{N-n} \leq t] - \mathbb{P}[W_{N-n+1} \leq t], \quad n = 1, 2, \ldots, N.$$ 

But we know from studying Poisson processes that

$$\mathbb{P}[W_k \leq t] = \sum_{j=k}^{\infty} e^{-\theta t} \left(\frac{\theta t}{j!}\right)^j.$$ 

Combining this with the last-displayed equation we find that

$$\mathbb{P}[X(t) = n] = e^{-\theta t} \left(\frac{(\theta t)^N}{(N-n)!}\right), \quad n = 1, 2, \ldots, N.$$ 

Similarly,

$$\mathbb{P}[X(t) = 0] = \mathbb{P}[W_N \leq t] = \sum_{j=0}^{\infty} e^{-\theta t} \left(\frac{\theta t}{j!}\right)^j.$$ 

**Pr. 6.2.3.** A glance at the picture on page 287 of the text should be enough to convince you that the area under the trajectory of the pure death process is

$$\sum_{k=1}^{N} k \cdot S_k.$$ 

Consequently, the desired expectation is

$$\sum_{k=1}^{N} \frac{k}{\mu_k}.$$