Ex. 6.3.1. \( \lambda_n = \lambda \) for \( n = 0, 1, 2, \ldots \), and \( \mu_n = n\mu \) for \( n = 1, 2, 3, \ldots \), where \( 1/\mu \) is the mean lifetime of a particle.

Ex. 6.3.3. Using formulas (6.30a)-(6.30d),

\[
P[V(t) = 1] = (1 - \pi)P_{01}(t) + \pi P_{11}(t)
= (1 - \pi)[\pi - \pi e^{-\tau t}] + \pi[\pi + (1 - \pi)e^{-\tau t}]
= \pi - \pi e^{-\tau t} - \pi^2 + \pi^2 e^{-\tau t} + \pi^2 + \pi e^{-\tau t} - \pi^2 e^{-\tau t}
= \pi
\]

Pr. 6.3.1. It is clear from the memoryless property of the exponential distribution that \( X(t) \) is a Markov chain, indeed a birth-death process with state space \( \{0, 1\} \). If \( X(t) = 0 \), then a transition to state 1 occurs at the next arrival time of the Poisson process; since such arrivals occur at rate \( \lambda \), we have \( \lambda_1 = \lambda \). If \( X(t) = 1 \) then a transition to state 0 occurs at the next arrival of the Poisson process, but only if \( \xi \) makes a \( 1 \to 0 \) transition when \( N \) jumps; this occurs at rate \( \lambda(1 - \alpha) \) (because \( \xi \) jumps from 1 to 0 with probability \( 1 - \alpha \)), so \( \mu_1 = \lambda(1 - \alpha) \).

Pr. 6.3.3. For \( 0 < s < t \), because \( (1 - \pi, \pi) \) is the stationary distribution,

\[
\mathbf{E}[V(s)V(t)] = P[V(s) = 1 = V(t)]
= P[V(s) = 1]P_{11}(t - s)
= \pi \left[ \pi + (1 - \pi)e^{-\tau(t-s)} \right]
= \pi^2 + \pi(1 - \pi)e^{-\tau(t-s)}.
\]

But also

\[
\mathbf{E}[V(s)] = P[V(s) = 1] = \pi = P[V(t) = 1] = \mathbf{E}[V(t)],
\]

so

\[
\text{Cov}[V(s), V(t)] = \pi(1 - \pi)e^{-\tau(t-s)}.
\]

Pr. 6.3.4. Since \( V(u) \) takes on only the two values 0 and 1, we have \( \mathbf{E}[V(u)] = P[V(u) = 1] \). Therefore,

\[
\mathbf{E}[V(u)] = P[V(u) = 1] = P_{01}(u) = \pi - \pi e^{-\tau u},
\]
so

\[ E\{S(t)\} = \int_0^t E\{V(u)\} \, du = \int_0^t [\pi - \pi e^{-\tau u}] \, du = \pi t - \pi \frac{1 - e^{-\tau t}}{\tau}. \]

**Ex. 6.4.2.** We have

\[
\theta_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \frac{\alpha^n N(N-1) \cdots (N-n+1)}{\beta^n 1 \cdot 2 \cdots n} = \left( \frac{N}{n} \right) \left( \frac{\alpha}{\beta} \right)^n.
\]

In view of the binomial theorem,

\[
\sum_{n=0}^{N} \theta_n = [1 + (\alpha/\beta)]^N,
\]

so the stationary distribution for this birth and death process is

\[
\pi_n = \left( \frac{N}{n} \right) \left( \frac{\alpha}{\alpha+\beta} \right)^n \left( \frac{\beta}{\alpha+\beta} \right)^{N-n}, \quad n = 0, 1, 2, \ldots, N,
\]

namely the binomial distribution with parameters \( N \) and \( p = \alpha/(\alpha + \beta) \).

**Ex. 6.4.4.** Let \( X(t) \) be the number of machines that are operational at time \( t \).

(a) The process \( X(t) \) is a birth-death process with state space \( \{0, 1, 2\} \) and rates: \( \lambda_0 = \lambda_1 = \lambda, \mu_1 = \mu, \mu_2 = 2\mu \). The stationary distribution is

\[
\pi = \begin{bmatrix}
\frac{2\mu^2}{2\mu^2 + 2\mu \lambda + \lambda^2} & \frac{2\mu \lambda}{2\mu^2 + 2\mu \lambda + \lambda^2} & \frac{\lambda^2}{2\mu^2 + 2\mu \lambda + \lambda^2}
\end{bmatrix}.
\]

In particular, the long run probability that no machines are running is

\[
\pi_0 = \frac{2\mu^2}{2\mu^2 + 2\mu \lambda + \lambda^2}.
\]

(b) If at most one machine can operate (and thus be subject to failure) the death rate \( \mu_2 \) must be changed to \( \mu_2 = \mu \). With this change the stationary distribution becomes

\[
\pi = \begin{bmatrix}
\frac{\mu^2}{\mu^2 + \mu \lambda + \lambda^2} & \frac{\mu \lambda}{\mu^2 + \mu \lambda + \lambda^2} & \frac{\lambda^2}{\mu^2 + \mu \lambda + \lambda^2}
\end{bmatrix}.
\]

The long run probability that no machine is running is now

\[
\pi_0 = \frac{\mu^2}{\mu^2 + \mu \lambda + \lambda^2}.
\]
Pr. 6.4.1. Since \( M = N = 5, R = 1, \lambda = 2, \) and \( \mu = 1, \) we have

\[
\mu_n = n, \quad n = 1, 2, 3, 4, 5,
\]

and

\[
\lambda_n = 2, \quad n = 0, 1, 2, 3, 4.
\]

Therefore,

\[
\theta_k = \frac{2^k}{k!}, \quad k = 0, 1, 2, 3, 4, 5,
\]

and

\[
\Theta = \sum_{k=0}^{5} \theta_k = 1 + 2 + 2 + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} = \frac{109}{15}.
\]

Thus

\[
\pi_k = \frac{15 \cdot 2^k}{109 \cdot k!}, \quad k = 0, 1, 2, \ldots, 5.
\]

In equilibrium we have:

(a) The average number of machines operating is

\[
\sum_{k=1}^{5} k \pi_k = \frac{15}{109} \left[ 2 + 4 + 4 + \frac{8}{3} + \frac{4}{3} \right] = \frac{15}{109} \cdot 14 = 1.93.
\]

(b) The equipment utilization is

\[
\frac{210}{109} \cdot \frac{1}{5} = \frac{42}{109} = 0.385 = 38.5\%.
\]

(c) The average idle repair capacity is

\[
\pi_5 = \frac{15 \cdot 2^5}{109 \cdot 5!} = \frac{15 \cdot 32}{109 \cdot 120} = \frac{4}{109} = 0.0367 = 3.67\%.
\]