6.4.3. Let $X(t)$ be the number of machines operating at time $t$. This is a birth-death process with state space $\{0, 1, 2, 3, 4, 5\}$ and rates

$$\lambda_n = .5, \ n = 0, 1, 2, 3, 4, \quad \mu_n = .20n, \ n = 1, 2, 3, 4, 5.$$  

We have

$$\theta_k = \frac{(5/2)^k}{k!}, \ k = 0, 1, 2, 3, 4, 5.$$  

The sum of these is

$$\sum_{k=0}^{5} \frac{(5/2)^k}{k!} = \frac{8963}{768} = 11.6706.$$  

In particular,

$$\pi_5 = \frac{480000}{6883584} = 0.069731117,$$

so the repairman is idle only about 7% of the time.

Pr. 6.4.4. Notice that the total number of links possible is 6. Therefore the state space of $X(t)$ is $\{0, 1, 2, 3, 4, 5, 6\}$.

(a) $\lambda_k = \alpha(6 - k), \ k = 0, 1, 2, 3, 4, 5,$ and $\mu_k = \beta k, \ k = 1, 2, 3, 4, 5, 6.$  

(b) By part (a),

$$\theta_k = \binom{6}{k} \left(\frac{\alpha}{\beta}\right)^k, \ k = 0, 1, 2, \ldots, 6.$$  

Consequently

$$\pi_k = \binom{6}{k} \left(\frac{\alpha}{\alpha + \beta}\right)^k \left(\frac{\beta}{\alpha + \beta}\right)^{6-k}, \ k = 0, 1, 2, \ldots, 6;$$

namely, the binomial distribution with parameters 6 and $\alpha/(\alpha + \beta)$.

Pr. 6.4.6. For the specified birth rates and death rates,

$$\theta_k = \frac{(\lambda/\mu)^k}{k!}, \ k = 0, 1, 2,$$

while

$$\theta_3 = \frac{(\lambda/\mu)^3}{4}.$$  

The total of these is

$$1 + \rho + \rho^2/2 + \rho^3/4,$$

where $\rho = \lambda/\mu$. The long run chance that the computer is fully loaded is $\pi_2 + \pi_3$, which is

$$\frac{\rho^2/2 + \rho^3/4}{1 + \rho + \rho^2/2 + \rho^3/4}.$$
Pr. 6.5.2. (a) Let $u_n$ be the probability of absorption in state 0 when $X(0) = n$. Evidently
\[ u_0 = 1, \quad u_5 = 0, \]
and
\[
\begin{align*}
u_1 & = (1/5)u_2 + (4/5) \\
u_2 & = (2/5)u_3 + (3/5)u_1 \\
u_3 & = (3/5)u_4 + (2/5)u_2 \\
u_4 & = (1/5)u_3.
\end{align*}
\]
Solving this system recursively, from the bottom up, we find that
\[
\begin{align*}
u_4 & = (1/5)u_3, & \nu_3 & = (5/11)u_2, & \nu_2 & = (11/15)u_1, & \nu_1 & = 15/16.
\end{align*}
\]
Back substituting we obtain
\[
\begin{align*}
u_1 & = 15/16, & \nu_2 & = 11/16, & \nu_3 & = 5/16, & \nu_4 & = 1/16.
\end{align*}
\]
In particular, $\nu_2 = 11/16$.

(b) Now let $w_n$ be the mean time to absorption if the initial state is $n$. Clearly $w_0 = w_5 = 0$.

By first-jump analysis,
\[
\begin{align*}
w_1 & = (1/5) + (1/5)w_2 \\
w_2 & = (1/5) + (2/5)w_3 + (3/5)w_1 \\
w_3 & = (1/5) + (3/5)w_4 + (2/5)w_2 \\
w_4 & = (1/5) + (1/5)w_3.
\end{align*}
\]
This system has a unique solution:
\[
\begin{align*}w_1 & = w_4 = 1/3, & w_2 = w_3 = 2/3.
\end{align*}
\]
In particular, the mean absorption time is $2/3$ when $X(0) = 2$.

Ex. 6.6.1. Each component of the system is a two-state Markov chain with infinitesimal matrix like the matrix at the foot of page 329 of the text, with $\alpha$ and $\beta$ specialized to either $\alpha_A$ and $\beta_A$ or $\alpha_B$ and $\beta_B$. The state space of the system is $S = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Write $\pi^A$ and $\pi^B$ for the stationary distribution for components $A$ and $B$ respectively. Thus, for example,
\[
\pi^A = \begin{bmatrix} \pi_0^A & \pi_1^A \end{bmatrix} = \begin{bmatrix} \frac{\beta_A}{\alpha_A + \beta_A} & \frac{\alpha_A}{\alpha_A + \beta_A} \end{bmatrix}.
\]
The stationary distribution for the system will be written as
\[
\pi = \begin{bmatrix} \pi_{00} & \pi_{01} & \pi_{10} & \pi_{11} \end{bmatrix},
\]
where I write 00 instead of $(0, 0)$, etc. We seek $1 - \pi_{00}$, the probability that at least one component is operating.
(a) Because the two components are independent, \( \pi(00) = \pi_0^A \pi_0^B = \frac{\beta_A \beta_B}{(\alpha_A + \beta_A)(\alpha_B + \beta_B)} \). Therefore

\[
1 - \pi(00) = 1 - \frac{\beta_A \beta_B}{(\alpha_A + \beta_A)(\alpha_B + \beta_B)} = \frac{\alpha_A \alpha_B + \alpha_A \beta_B + \beta_A \beta_B}{(\alpha_A + \beta_A)(\alpha_B + \beta_B)}.
\]

Therefore

\[
1 - \pi(00) = 1 - \frac{\beta_A \beta_B}{(\alpha_A + \beta_A)(\alpha_B + \beta_B)} = \pi_0^A \pi_0^B (\alpha_A + \beta_A)(\alpha_B + \beta_B).\]

(b) Now consider the system as a four-state Markov chain with state space \( S \) as above and one-step transition matrix

\[
P = \begin{bmatrix}
-(\alpha_A + \alpha_B) & \alpha_B & \alpha_A & 0 \\
\beta_B & -(\alpha_A + \beta_B) & 0 & \alpha_A \\
\alpha_B & 0 & -(\alpha_B + \beta_B) & \beta_A \\
0 & \beta_A & \beta_B & -(\beta_A + \beta_B)
\end{bmatrix}.
\]

The stationary distribution \( \pi^A = 0 \), written in detail, is the system

\[
\begin{align*}
-\pi(00) (\alpha_A + \alpha_B) + \pi(01) \beta_B + \pi(10) \alpha_B &= 0 \\
\pi(00) \alpha_B - \pi(01) (\alpha_A + \beta_B) + \pi(11) \beta_A &= 0 \\
\pi(00) \alpha_A - \pi(10) (\alpha_B + \beta_A) + \pi(11) \beta_B &= 0 \\
\pi(01) + \pi(10) - \pi(11) (\beta_A + \beta_B) &= 0.
\end{align*}
\]

Rather than writing out the solution in full, let me indicate the steps one can take. Solve equation 2 for \( \pi(01) \) in terms of \( \pi(00) \) and \( \pi(11) \); likewise, solve equation 3 for \( \pi(10) \) in terms of \( \pi(00) \) and \( \pi(11) \). Substitute these expression for \( \pi(01) \) and \( \pi(10) \) into equations 1 and 4, thereby obtaining a system of two equations in the two unknowns \( \pi(00) \) and \( \pi(11) \). Solving this two-by-two system we obtain the expression for \( \pi(00) \) found in part (a).

Pr. 6.6.2. For \( i = A, B, C \), let \( \pi_i^1 \) be the stationary probability of state 1 for component \( i \). Thus,

\[
\pi_i^1 = \frac{\alpha_i}{\alpha_i + \beta_i}.
\]

With this notation, the long run probability that the system is operating is

\[
\pi_1^A \left[ \pi_1^B + \pi_1^C - \pi_1^B \pi_1^C \right].
\]

Pr. 6.6.4. We list the states in the order given in the statement of the problem:

\[
(2, 0), (1, 0), (1, 1), (0, 0), (0, 1), (0, 2).
\]

With this convention, the infinitesimal matrix is

\[
A = \begin{bmatrix}
-2\mu & 2p\mu & 2q\mu & 0 & 0 & 0 \\
\alpha & -\alpha - \mu & 0 & p\mu & q\mu & 0 \\
\beta & 0 & -\beta - \mu & 0 & p\mu & q\mu \\
0 & \alpha & 0 & -\alpha & 0 & 0 \\
0 & \beta & \alpha & 0 & -\alpha - \beta & 0 \\
0 & 0 & \beta & 0 & 0 & -\beta
\end{bmatrix}.
\]