Math 180C, Spring 2019

Homework 4 Solutions

Ex. 7.1.2. We have

\[ F_2(t) = \int_0^t F(t - s)f(s) \, ds \]
\[ = \int_0^t (1 - e^{-\lambda(t-s)})\lambda e^{-\lambda s} \, ds \]
\[ = \int_0^t (\lambda e^{-\lambda s} - \lambda e^{-\lambda t}) \, ds \]
\[ = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}. \]

Therefore,

\[ P[N(t) = 1] = P[W_1 \leq t < W_2] = P[W_1 \leq t] - P[W_2 \leq t] \]
\[ = 1 - e^{-\lambda t} - 1 + e^{-\lambda t} + \lambda t e^{-\lambda t} \]
\[ = \lambda t e^{-\lambda t}. \]

Ex. 7.1.3. (a) True. By complements this is the same as: \( N(t) \geq k \) if and only if \( W_k \leq t \).

(b) False. \( N(t) \leq k \) if and only if \( N(t) < k + 1 \) if and only if \( W_{k+1} > t \). Thus, if \( W_k < t < W_{k+1} \) (a distinct possibility) the assertion fails.

(c) False. Take complements in (b).

Pr. 7.1.1. The first equality follows because \( \delta_t \geq x \) and \( \gamma_t > y \) if and only if there are no renewals in the time interval \( (t - x, t + y) \). The second equality results from the law of total probability—\( k \) is running through the possible values of \( N(t - x) \). That is

\[ P[N(t - x) = N(t + y)] = \sum_{k=0}^{\infty} P[N(t - x) = k = N(t + y)]. \]

The \( k^{th} \) term in the above sum is

\[ P[N(t - x) = k = N(t + y)] = P[W_k \leq t - x, W_{k+1} > t + y]. \]

For the third equality, split off the \( k = 0 \) term by itself, and condition on the value of \( W_k \) in each of the remaining terms. Then, for \( k \geq 1 \),

\[ P[W_k \leq t - x, W_{k+1} > t + y] = \int_0^{t-x} P[W_{k+1} - W_k > t + y - s] f_k(s) \, ds. \]
\[ = \int_0^{t-x} [1 - F(t + y - s)] f_k(s) \, ds. \]
When the inter-occurrence times are exponential, the density \( f_k \) is the gamma density as noted in the text. In this case we can sum the series:

\[
\sum_{k=1}^{\infty} \int_{0}^{t-x} [1 - F(t + y - s)] f_k(s) \, ds = \int_{0}^{t-x} [1 - F(t + y - s)] \left[ \sum_{k=1}^{\infty} \frac{\lambda^k s^{k-1}}{(k-1)!} e^{-\lambda s} \right] ds
\]

\[
= \int_{0}^{t-x} [1 - F(t + y - s)] \lambda e^{-\lambda s} \left[ \sum_{j=0}^{\infty} \frac{\lambda^j s^j}{j!} \right] ds
\]

\[
= \int_{0}^{t-x} \lambda e^{-\lambda(t+y-s)} ds
\]

\[
= e^{-\lambda(t+y)} - e^{-\lambda(t+y)}.
\]

Adding this to the \( k = 0 \) term we obtain

\[
P[\delta_t \geq x, \gamma_t > y] = e^{-\lambda(x+y)},
\]

provided \( t > x \).

**Pr. 7.1.3.** Because \( \gamma_t = W_{N(t)+1} - t \),

\[
E[\gamma_t] = E[W_{N(t)+1}] - t = \mu[M(t) + 1] - t.
\]

**Ex. 7.2.2.** The separate components \( A \) and \( B \) of the system behave like independent Markov chains \( X_A(t) \) and \( X_B(t) \) with state space \( \{0, 1\} \) and infinitesimal matrices

\[
A_A = \begin{bmatrix} 1 - \alpha_A & \alpha_A \\ \beta_A & 1 - \beta_A \end{bmatrix}
\]

and

\[
A_B = \begin{bmatrix} 1 - \alpha_B & \alpha_B \\ \beta_B & 1 - \beta_B \end{bmatrix}.
\]

The pair \( X(t) = (X_A(t), X_B(t)) \) is then a Markov chain with state space

\[
\{(0, 0), (0, 1), (1, 0), (1, 1)\},
\]

and the *system* renews itself when \( X(t) \) moves from one of the states \( (1, 0), (0, 1) \) to the state \( (0, 0) \). By the continuous time analog of example (f) on page 354, the times of return to state \( (0, 0) \) form a renewal process.