Math 280B, Winter 2005

Doob's Inequalities

Everything that follows takes place on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_n : n = 0, 1, 2, \ldots\}$, with $\mathcal{F}_n \subset \mathcal{F}$ for all n.

1. Submartingale maximal inequality. *Let $\{X_n\}$ be a non-negative submartingale (for example, $X_n = |M_n|$ if $\{M_n\}$ is a martingale, or $X_n = S_n^+$ if $\{S_n\}$ is a submartingale), and define $X_n^* := \max_{0 \le k \le n} X_k$. Then*

$$P[X_n^* \ge t] \le t^{-1} E[X_n; X_n^* \ge b] \le t^{-1} E[X_n], \qquad \forall t > 0.$$

For the proof of this maximal inequality we require the following simple lemma, a hint of better things.

2. Lemma. *If $\{Y_n\}$ is a submartingale and T is a stopping time bounded above by a positive integer N, then*

$$Y_T \le E[Y_N | \mathcal{F}_T].$$

Proof. If $A \in \mathcal{F}_T$, then

$$\begin{aligned} E[Y_N; A] &= \sum_{n=0}^{N} E[Y_N; A \cap \{T = n\}] \ge \sum_{n=0}^{N} E[Y_n; A \cap \{T = n\}] \\ &= \sum_{n=0}^{N} E[Y_T; A \cap \{T = n\}] = E[Y_T; A], \end{aligned}$$

where the inequality follows from the submartingale property of Y because $A \cap \{T = n\} \in \mathcal{F}_n$. $\square$

3. Proof of the maximal inequality. Fix a positive integer n and define $T := \min\{k \ge 0 : X_k \ge b\} \wedge n$. Then T is a stopping time bounded above by n and

$$\{X_n^* \ge b\} = \{X_T \ge b\}.$$

Thus,

$$\begin{aligned} P[X_n^* \ge t] &= P[X_T \ge t] \le E[X_T/t; X_T \ge t] \\ &\le t^{-1} E[X_n; X_T \ge t] = t^{-1} E[X_n; X_n^* \ge t] \\ &\le t^{-1} E[X_n], \end{aligned}$$

the second inequality following from the Lemma. $\square$

Doob's L^p maximal inequality is a corollary of the submartingale maximal inequality. The proof is based on the following calculation (extending one seen in Math 280A), which is a simple consequence of Tonelli's theorem.

4. Lemma. *Let W and Z be non-negative random variables. Then for any $r > 0$,*

$$E[W \cdot Z^r] = r \int_0^\infty t^{r-1} E[W; Z > t]\, dt.$$

5. L^p Maximal Inequality. *If $\{X_n\}$ is a positive submartingale and $1 < p < \infty$, then for $n = 0, 1, 2, \ldots$,*

$$\|X_n^*\|_p \le C_p \|X_n\|_p,$$

where $X_n^* := \max_{0\le k\le n} X_k$ and $C_p := p/(p-1)$.

Proof. Fix n. By Lemma **4** (twice) and the maximal inequality **1**,

$$\begin{aligned} E[(X_n^*)^p] &= p \int_0^\infty t^{p-1} P[X_n^* > t]\, dt \\ &\le p \int_0^\infty t^{p-2} P[X_n; X_n^* > t]\, dt \\ &= \frac{p}{p-1} E[X_n (X_n^*)^{p-1}]. \end{aligned}$$

Thus, by Hölder's inequality,

$$\|X_n^*\|_p^p = E[(X_n^*)^p] \le \frac{p}{p-1} E[X_n (X_n^*)^{p-1}] \le C_p^p \|X_n\|_p \cdot \|(X_n^*)^{p-1}\|_q. \tag{1}$$

Here $q = p/(p-1)$ is the conjugate exponent of p. In particular, $(p-1)q = p$, so $\|(X_n^*)^{p-1}\|_q = \|X_n^*\|_p^{p/q}$. Therefore, (1) implies

$$\|X_n^*\|_p^{p-p/q} \le C_p \|X_n\|_p,$$

which is the stated inequality because $p - p/q = 1$. ☐

6. Submartingale upcrossing inequality. *Let $\{X_n\}$ be a submartingale, and for real numbers $a < b$ let $U_n = U_n(a,b)$ by the number of upcrossings of the interval (a,b) that X completes by time n. Then for $n = 1, 2, \ldots$,*

$$E[U_n] \le \frac{E[(X_n - a)^+] - E[(X_0 - a)^+]}{b-a}.$$

Proof. [The proof is the additive analog of the proof of Dubins's inequality.] Define recursively, $T_1 = \min\{k \ge 0 : X_k \le a\}$, $T_2 = \min\{k \ge T_1 : X_k \ge b\}$, $T_3 = \min\{k \ge T_2 : X_n \le a\}$, etc. Then

$$\{U_n \ge m\} = \{T_{2m} \le n\},$$

and (just as in the discussion of Dubins's inequality)

$$H_k := \begin{cases} 1 & \text{if } T_{2m-1} < k \le T_{2m} \text{ for some } m \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

defines a bounded predictable process. Notice that the process $Y_n := (X_n - a)^+$ is a non-negative submartingale, and that the number of upcrossings of (a,b) by X is precisely the same as the number of upcrossings of $(0, b-a)$ by Y. Also, $\{((1-H)\cdot Y)_n\}$ is a submartingale because $0 \le H_k \le 1$. Because each upcrossing completed by time n contributes at least $b-a$ to the total determining $(H\cdot Y)_n$, and the possible upcrossing-in-progress at time n contributes a non-negative amount, we have

$$(b-a)U_n \le (H\cdot Y)_n.$$

Thus,

$$\begin{aligned}(b-a)E[U_n] &\le E[(H\cdot Y)_n] = E[Y_n - ((1-H)\cdot Y)_n] \\ &\le E[Y_n] - E[((1-H)\cdot Y)_0] \\ &= E[(X_n-a)^+] - E[(X_0-a)^+].\end{aligned}$$

□

7. Corollary. *If $\{X_n\}$ is a submartingale with $\sup_n E[X_n^+] < \infty$, then $X_\infty := \lim_n X_n$ exists almost surely, and X_∞ is integrable.*

Proof. Suppose that $M := \sup_n E[X_n^+] < \infty$. From the elementary inequality $(x-a)^+ \le x^+ + a^-$ we deduce that for any real a

$$E[(X_n-a)^+] \le M + a^-$$

for all n. Thus, by Fatou's lemma, the total number $U_\infty(a,b) :=\uparrow \lim_n U_n(a,b)$ of upcrossings of (a,b) made by X has finite expectation:

$$E[U_\infty(a,b)] \le \liminf_n \frac{E[(X_n-a)^+]}{b-a} \le M + |a| < \infty.$$

In particular,

$$P[U_\infty(a,b) < \infty] = 1, \qquad \forall a < b.$$

Therefore $X_\infty := \lim_n X_n$ exists almost surely. Moreover, $E[X_\infty^+] \le M < \infty$ by Fatou. On the other hand, because $x^- = x^+ - x$, Fatou's lemma also yields

$$E[X_\infty^-] \le \liminf_n E[X_n^-] = \liminf_n E[X_n^+ - X_n] \le \liminf_n E[X_n^+ - X_0] \le M - E[X_0] < \infty,$$

where the second inequality follows from the submartingale property of X. It follows that $E|X_\infty| < \infty$; in particular, X_∞ is finite almost surely. □