Math 280B, Winter 2005

SLLN for Martingales

Everything that follows takes place on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a filtration $\{\mathcal{F}_n : n = 0, 1, 2, \ldots\}$, with $\mathcal{F}_n \subset \mathcal{F}$ for all n.

1. Square-integrable martingales. A martingale $M = (M_n)$ is said to be *square integrable* if $\mathbf{E}[M_n^2] < \infty$ for each n. In this case M_n^2 is a submartingale, with Doob decomposition

$$M_n^2 = M_0^2 + X_n + \langle M \rangle_n, \qquad n \geq 0,$$

where (X_n) is a martingale with $X_0 = 0$ and the (predictable) *quadratic variation* process $\langle M \rangle$ is defined by $\langle M \rangle_0 = 0$ and

$$\langle M \rangle_n := \sum_{k=1}^{n} \mathbf{E}[(\Delta M_k)^2 | \mathcal{F}_{k-1}], \qquad n \geq 1. \tag{1}$$

A variation of a stopping time argument used in class proves the following result. We write $\langle M \rangle_\infty := \lim_n \langle M \rangle_n$, and $\{M_n \to\}$ for the event on which (M_n) converges to a real-valued limit.

2. Theorem. *Let $M = (M_n)$ be a square-integrable martingale with quadratic variation $\langle M \rangle$ as defined in (1).*

(a) $\{\langle M \rangle_\infty < \infty\} \subset \{M_n \to\}$, a.s.

(b) If $\mathbf{E}[\sup_n (\Delta M_n)^2] < \infty$, then $\{\langle M \rangle_\infty < \infty\} = \{M_n \to\}$, a.s.

Proof. (a) For $b > 0$ define $T_b := \inf\{n : \langle M \rangle_{n+1} > b\}$. Then T_b is a stopping time (because $\langle M \rangle$ is predictable), and $\langle M \rangle_{T_b} \leq b$. The stopped process $X_{T_b \wedge n}$ is a martingale, so

$$\mathbf{E}[M_{T_b \wedge n}^2] = \mathbf{E}[X_0] + \mathbf{E}[\langle M \rangle_{T_b \wedge n}] = \mathbf{E}[M_0^2] + \mathbf{E}[\langle M \rangle_{T_b \wedge n}] \leq \mathbf{E}[M_0^2] + b.$$

That is, the stopped martingale $(M_{T_b \wedge n})$ is L^2-bounded. Thus, $(M_{T_b \wedge n})$ converges a.s., so (M_n) converges a.s. on the event $\{T_b = \infty\}$. Varying b over the positive integers, we see that (M_n) converges a.s. on the event $\cup_{b \in \mathbf{N}} \{T_b = \infty\} = \{\langle M \rangle_\infty < \infty\}$.

(b) Define, for $b > 0$, stopping times $S_b := \inf\{n : M_n^2 > b\}$, and note that $|M_{S_b - 1}| \leq \sqrt{b}$. Consequently, on the event $\{S_b < \infty\}$,

$$\begin{aligned} |\Delta(M_{S_b}^2)| = |M_{S_b}^2 - M_{S_b - 1}^2| &\leq (\Delta M_{S_b})^2 + 2|M_{S_b - 1}| \cdot |\Delta M_{S_b}| \\ &\leq (\Delta M_{S_b})^2 + 2\sqrt{b} \cdot |\Delta M_{S_b}|. \end{aligned}$$

Thus,

$$\mathbf{E}[|\Delta(M_{S_b}^2)|; S_b < \infty] \le C + 2\sqrt{b} + \sqrt{C} < \infty,$$

where $C := \mathbf{E}[\sup_n (\Delta M_n)^2] < \infty$. Now, because $M_n^2 - \langle M \rangle_n$ is a martingale,

$$\mathbf{E}[\langle M \rangle_{S_b \wedge n}] = \mathbf{E}[M_{S_b \wedge n}^2] - \mathbf{E}[M_0^2] \le \mathbf{E}[M_{S_b \wedge n}^2] \le b^2 + \mathbf{E}[|\Delta(M_{S_b}^2)|; S_b < \infty] < \infty,$$

and so, by Fatou's lemma,

$$\mathbf{E}[\langle M \rangle_{S_b}] \le b^2 + \mathbf{E}[|\Delta(M_{S_b}^2)|; S_b < \infty] < \infty.$$

Thus $\langle M \rangle_{S_b} < \infty$ a.s. In particular, $\langle M \rangle_\infty < \infty$ on $\{S_b = \infty\}$. Varying $b > 0$ we see that $\langle M \rangle_\infty < \infty$ on the event $\cup_{b=1}^\infty \{S_b = \infty\} = \{\sup_n M_n^2 < \infty\} \supset \{M_n \to\}$. ∎

As a corollary of Theorem 2 we have the following Strong Law of Large Numbers for square-integrable martingales.

3. Theorem. *Let $M = (M_n)$ be a square-integrable martingale with quadratic variation $\langle M \rangle$ as defined in (1). Then*

$$\frac{M_n}{\langle M \rangle_n} \to 0 \qquad \text{a.s. on the event } \{\langle M \rangle_\infty = \infty\}.$$

Proof. On $\{\langle M \rangle_\infty = \infty\}$, we have

$$\frac{\langle M \rangle_n}{1 + \langle M \rangle_n} \to 1 \text{ as } n \to \infty.$$

It therefore suffices to show that, a.s. on $\{\langle M \rangle_\infty = \infty\}$,

$$\frac{M_n}{1 + \langle M \rangle_n} \to 0. \tag{2}$$

In view of Kronecker's lemma, to show (2) it suffices to show that

$$K_n := \sum_{k=1}^n \frac{\Delta M_k}{1 + \langle M \rangle_k}$$

converges as $n \to \infty$, a.s. on $\{\langle M \rangle_\infty = \infty\}$. But (K_n) is a square-integrable martingale with

$$\langle K \rangle_n = \sum_{k=1}^n \frac{\mathbf{E}[(\Delta M_k)^2 | \mathcal{F}_{k-1}]}{(1 + \langle M \rangle_k)^2} \le \sum_{k=1}^n (\Delta M_k)^2 = \langle M \rangle_n.$$

By part (a) of Theorem 2, $\{\langle K\rangle_\infty < \infty\} \subset \{K_n \to\}$ (a.s.). To conclude, we must verify that $\langle K\rangle_\infty < \infty$. To this end notice that

$$\langle K\rangle_\infty = \sum_{k=1}^{\infty} \frac{\Delta\beta_k}{\beta_k^2},$$

where $\beta_k := 1 + \langle M\rangle_k > 0$, so that $\Delta\beta_k \geq 0$. Because

$$-\Delta\left(\frac{1}{\beta_k}\right) = -\left(\frac{1}{\beta_k} - \frac{1}{\beta_{k-1}}\right) = \frac{\Delta\beta_k}{\beta_k\beta_{k-1}} \geq \frac{\Delta\beta_k}{\beta_k^2},$$

we have

$$\sum_{k=1}^{n} \frac{\Delta\beta_k}{\beta_k^2} \leq -\sum_{k=1}^{n} \Delta(1/\beta_k) = \beta_0^{-1} - \beta_n^{-1} = 1 - \beta_n^{-1} \leq 1.$$

It follows that $\langle K\rangle_\infty \leq 1$ a.s., and we are done. $\square$

The SLLN in hand, it is quite natural to look for martingale results analogous to the CLT. The following basic martingale CLT is due to B.M. Brown [*Annals of Math. Stat.* **42** (1971) 59–66], to which you are referred for the proof.

4. Theorem. *Let $M = (M_n)$ be a square-integrable martingale with quadratic variation $\langle M\rangle$ as defined in (1). Define $s_n^2 := \mathbf{E}[\langle M\rangle_n]$. Assume that*

$$\frac{\langle M\rangle_n}{s_n^2} \xrightarrow{P} 1$$

and that the Lindeberg-type condition

$$s_n^{-2} \sum_{k=1}^{n} \mathbf{E}[(\Delta M_k)^2; (\Delta M_k)^2 \geq \epsilon \cdot s_n] \xrightarrow{P} 0, \qquad \forall \epsilon > 0,$$

is satisfied. Then $M_n/s_n \Rightarrow \mathcal{N}(0,1)$.