

Polar Coordinates

Math 280B, Winter 2005

We write

$$B^d(r) := \{x \in \mathbf{R}^d : |x| < r\}$$

for the (open) ball of radius $r > 0$ (centered at the origin) in \mathbf{R}^d , and

$$S^{d-1}(r) := \{x \in \mathbf{R}^d : |x| = r\}$$

for its boundary, the sphere on radius r in \mathbf{R}^d . Let λ_d denote Lebesgue measure on the Borel subsets $\mathcal{B}(\mathbf{R}^d)$ of d -dimensional Euclidean space \mathbf{R}^d , and let σ_{d-1} denote the surface measure on the unit sphere $S^{d-1} = S^{d-1}(1)$. We write Ω_d for the volume of the unit ball $B^d = B^d(1)$ in \mathbf{R}^d and ω_d for $\sigma_{d-1}(S^{d-1})$.

(1) Polar Coordinates Formula. *If $f : \mathbf{R}^d \rightarrow \mathbf{R}$ is Borel measurable and Lebesgue integrable (or non-negative), then*

$$(2) \quad \int_{\mathbf{R}^d} f(x) \lambda_d(dx) = \int_0^\infty \int_{S^{d-1}} f(ru) \sigma_{d-1}(du) r^{d-1} dr.$$

Corollary.

$$(3) \quad d \cdot \Omega_d = \omega_{d-1}.$$

Proof. Take $f = 1_{B^d}$ in (2). Then $f(ru) = 1_{[0,1]}(r)$ for all $u \in S^{d-1}$, and so

$$\Omega_d = \int_0^1 r^{d-1} dr \cdot \omega_{d-1} = d^{-1} \cdot \omega_{d-1}.$$

□

More generally,

$$\lambda_d[B^d(r)] = \Omega_d \cdot r^d,$$

and

$$\sigma_d[S^{d-1}(r)] = \omega_{d-1} \cdot r^{d-1}.$$

In particular,

$$\frac{d}{dr} \lambda_d[B^d(r)] = \sigma_{d-1}[S^{d-1}(r)], \quad r > 0.$$

Next, take $f(x) = \exp(-|x|^2/2)$ in (2). As is well known, the left side of (2) is then equal to $(2\pi)^{d/2}$. Meanwhile the right side is

$$\omega_{d-1} \cdot \int_0^\infty e^{-r^2/2} r^{d-1} dr = 2^{d/2-1} \omega_{d-1} \cdot \int_0^\infty e^{-t} t^{d/2-1} dt = 2^{d/2-1} \omega_{d-1} \Gamma(d/2).$$

But

$$2^{d/2-1} \omega_{d-1} \Gamma(d/2) = 2^{d/2} \Omega_d \Gamma(d/2 + 1),$$

So

$$(4) \quad \Omega_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}$$

and

$$(5) \quad \omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

This formula gives

$$\omega_1 = 2\pi, \quad \omega_2 = 4\pi, \quad \omega_3 = 2\pi^2,$$

as expected.

By Stirling's formula,

$$\Gamma(t+1) \sim \sqrt{2\pi t} \left(\frac{t}{e}\right)^t, \quad t \rightarrow \infty,$$

so

$$\Omega_d \sim \frac{1}{\sqrt{\pi d}} \left(\frac{2\pi e}{d}\right)^{d/2}, \quad d \rightarrow \infty.$$

For example, $\Omega_{10} \approx 2.59$, which is a tiny fraction (approximately 0.25 %) of the volume of the smallest cube in \mathbf{R}^{10} containing B^{10} .