Math 294, Winter 2006

Homework 4 — Due March 15

In problems **1** and **2**, $W = (W_t)_{0\le t\le T}$ is a (standard) Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$, and $\mathcal{F} = \mathcal{F}_T$.

1. Let μ and ν be real numbers. With W as described above, the process $X_t := W_t + \mu t$ is a *Brownian motion with drift* μ. Using Girsanov's theorem, find an equivalent probability measure $\mathbf{Q}$ (on $(\Omega, \mathcal{F}_T)$) under which $(X_t)_{0\le t\le T}$ is a Brownian motion with drift ν. That is, $X_t = \widetilde{W}_t + \nu t$, $0 \le t \le T$, where $\widetilde{W}_t := W_t + (\mu - \nu)t$ is a $\mathbf{Q}$-Brownian motion.

2. Consider, in the Black-Scholes setup, a European contingent claim (called a *forward start option*) whose payoff at time T is

$$X = (S_T - S_{T_0})^+,$$

where $T_0 \in (0, T)$ is a fixed time. This is like a European call option, except that the strike price is (the random variable) S_{T_0}. Determine the no-arbitrage time-zero price of this option X. [Hint: Recall that

$$S^*_T = S^*_{T_0} \exp\Big(\sigma(\widetilde{W}_T - \widetilde{W}_{T_0}) - \sigma^2(T - T_0)/2\Big)$$

and that $\widetilde{W}_T - \widetilde{W}_{T_0}$ is independent of $\mathcal{F}_{T_0}$ (in particular, independent of $S^*_{T_0}$). Thus

$$X^* = \Big(S^*_T - S^*_{T_0} e^{-r(T-T_0)}\Big)^+ = S^*_{T_0} \cdot \Big(\widehat{S}^*_{T-T_0} - e^{-r(T-T_0)}\Big)^+,$$

where $\widehat{S}^*_{T-T_0} = \exp\Big(\sigma(\widetilde{W}_T - \widetilde{W}_{T_0}) - \sigma^2(T - T_0)/2\Big)$.]

3. Section 4.10, Exercise 3

4. Section 4.10, Exercise 4