

## Math 294, Winter 2006

### Lévy's Theorem

Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a complete probability space endowed with a right-continuous\* filtration  $(\mathcal{F}_t)_{t \geq 0}$  such that  $\mathcal{F}_0$  contains all the  $\mathbf{P}$ -null sets in  $\mathcal{F}$  and  $\bigvee_t \mathcal{F}_t = \mathcal{F}$ . Let  $M = (M_t)_{t \geq 0}$  be a real-valued stochastic process adapted to  $(\mathcal{F}_t)$  with continuous sample paths. We assume that  $M_0 = 0$ .

**Theorem.** *Suppose that both  $M$  and  $(M_t^2 - t)_{t \geq 0}$  are local martingales. Then  $M$  is a Brownian motion with respect to  $(\mathcal{F}_t)$ . More precisely, if  $0 < s < t$ , then  $M_t - M_s$  is independent of  $\mathcal{F}_s$  and is normally distributed with mean 0 and variance  $t - s$ .*

*Proof.* The key observation (due to H. Kunita & S. Watanabe) is that the development of the Itô integral (and Itô's formula) for Brownian motion  $(W_t)$  rests solely on the fact that  $W_t$  and  $W_t^2 - t$  are (local) martingales. It follows that if  $f \in C^2(\mathbf{R})$  then

$$(1) \quad f(M_t) = f(0) + \int_0^t f'(M_s) dM_s + \frac{1}{2} \int_0^t f''(M_s) ds,$$

where the stochastic integral  $M_t^f := \int_0^t f'(M_s) dM_s$  is a local martingale. In particular, if  $f'$  is bounded then  $M_t^f$  is a martingale, in which case upon taking expectations in (1) we obtain

$$(2) \quad \mathbf{E}[f(M_t)] = f(0) + \frac{1}{2} \int_0^t \mathbf{E}[f''(M_s)] ds.$$

Let us take  $f$  in (2) to be of the form  $f(x) = \exp(i\theta x)$ , where  $\theta \in \mathbf{R}$  and  $i = \sqrt{-1}$ . Writing  $g(t) := \mathbf{E}[\exp(i\theta M_t)]$  we obtain

$$g(t) = 1 - \frac{\theta^2}{2} \int_0^t g(s) ds$$

because  $f''(x) = -\theta^2 f(x)$ . Consequently,  $g$  satisfies the initial value problem

$$g'(t) = -\frac{\theta^2}{2} g(t) \quad g(0) = 1,$$

which has the unique solution  $g(t) = \exp(-t\theta^2/2)$ . Thus

$$\mathbf{E}[\exp(i\theta M_t)] = \exp(-t\theta^2/2), \quad \theta \in \mathbf{R},$$

---

\* *i.e.*,  $\mathcal{F}_t = \mathcal{F}_{t+}$  for all  $t \geq 0$ .

which means that  $M_t \sim \mathcal{N}(0, t)$ .

Now fix  $s > 0$  and  $A \in \mathcal{F}_s$  with  $\mathbf{P}(A) > 0$ . Define  $\mathbf{P}^*(B) := \mathbf{P}(B \cap A)/\mathbf{P}(A) = \mathbf{P}(B|A)$ ,  $\mathcal{F}_t^* := \mathcal{F}_{t+s}$ , and  $M_t^* := M_{t+s} - M_s$  for  $t \geq 0$ . Then with respect to the filtration  $(\mathcal{F}_t^*)$  over the probability space  $(\Omega, \mathcal{F}, \mathbf{P}^*)$ , the stochastic process  $(M_t^*)_{t \geq 0}$  is a continuous local martingale with  $M_0^* = 0$  such that  $[M_t^*]^2 - t$  is also a local martingale. The considerations of the preceding paragraph apply to this process, and we deduce that

$$(3) \quad \mathbf{E}^*[\exp(i\theta M_t^*)] = \exp(-t\theta^2/2).$$

Writing the “starred” objects explicitly, (3) becomes

$$(4) \quad \mathbf{E}[\exp(i\theta(M_{t+s} - M_t)); A] = \exp(-t\theta^2/2)\mathbf{P}(A).$$

Varying  $A \in \mathcal{F}_s$  in (4) we find that

$$\mathbf{E}[\exp(i\theta(M_{t+s} - M_t))|\mathcal{F}_s] = \exp(-t\theta^2/2),$$

which shows that  $M_{t+s} - M_s$  is independent of  $\mathcal{F}_s$  and has the  $\mathcal{N}(0, t)$  distribution.  $\square$

**Example 1.** Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$  be a filtered probability space, and let  $W = (W_t)_{t \geq 0}$  be an  $(\mathcal{F}_t)$  Brownian motion. Let  $H = (H_t)_{t \geq 0}$  be a measurable  $(\mathcal{F}_t)$  adapted process taking on only the two values  $\pm 1$ . Then  $H \in \mathcal{L}^2$  so the stochastic integral

$$M_t := \int_0^t H_s dW_s, \quad t \geq 0,$$

is a square-integrable martingale. Moreover,  $\langle M \rangle_t = \int_0^t H_s^2 ds = \int_0^t 1 ds = t$ , so  $M_t^2 - t$  is also a local martingale. It follows from Lévy’s theorem that  $M$  is also an  $(\mathcal{F}_t)$  Brownian motion.

**Example 2.** As in the previous example, let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$  be a filtered probability space, and let  $W = (W_t)_{t \geq 0}$  be an  $(\mathcal{F}_t)$  Brownian motion. Now let  $H = (H_t)_{t \geq 0}$  be an arbitrary element of  $\mathcal{L}_{\text{loc}}^2$ . As before we define the local martingale

$$M_t := \int_0^t H_s dW_s, \quad t \geq 0,$$

which has quadratic variation process

$$\langle M \rangle_t = \int_0^t H_s^2 ds, \quad t \geq 0.$$

Observe that  $\langle M \rangle_t$  is continuous and non-decreasing. Let us assume that, almost surely,

$$(5) \quad \lim_{t \rightarrow \infty} \langle M \rangle_t = \infty.$$

Define

$$T(s) := \inf\{t : \langle M \rangle_t > s\}, \quad s \geq 0.$$

Then (5) implies that  $T(s)$  is finite (a.s.) for each  $s \geq 0$ , and it is not hard to check that each  $T(s)$  is a stopping time. We use these stopping times to “time change”  $M$  into Brownian motion. Precisely, define

$$C_s := M_{T(s)}, \quad s \geq 0,$$

and

$$\mathcal{G}_s := \mathcal{F}_{T(s)}, \quad s \geq 0.$$

Then the stochastic process  $C = (C_s)_{s \geq 0}$  is adapted to the filtration  $(\mathcal{G}_s)_{s \geq 0}$ . The optional stopping theorem implies that  $C$  is a local martingale (with respect to  $(\mathcal{G}_s)$ ). Moreover, the quadratic variation interpretation of  $\langle C \rangle$  and  $\langle M \rangle$  implies that

$$\langle C \rangle_s = \langle M \rangle_{T(s)} = s, \quad \forall s \geq 0,$$

almost surely. That is,  $C_s^2 - s$  is also a  $(\mathcal{G}_s)$  local martingale. Lévy’s theorem now tells us that  $C = (C_s)$  is a  $(\mathcal{G}_s)$  Brownian motion. The moral: A continuous local martingale is just Brownian motion with its “clock” running too fast (or too slow).