Math 294, Winter 2006

Lévy's Theorem

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space endowed with a right-continuous* filtration $(\mathcal{F}_t)_{t\ge 0}$ such that $\mathcal{F}_0$ contains all the $\mathbf{P}$-null sets in $\mathcal{F}$ and $\vee_t \mathcal{F}_t = \mathcal{F}$. Let $M = (M_t)_{t\ge 0}$ be a real-valued stochastic process adapted to $(\mathcal{F}_t)$ with continuous sample paths. We assume that $M_0 = 0$.

Theorem. *Suppose that both M and $(M_t^2 - t)_{t\ge 0}$ are local martingales. Then M is a Brownian motion with respect to $(\mathcal{F}_t)$. More precisely, if $0 < s < t$, then $M_t - M_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and variance $t - s$.*

Proof. The key observation (due to H. Kunita & S. Watanabe) is that the development of the Itô integral (and Itô's formula) for Brownian motion (W_t) rests solely on the fact that W_t and $W_t^2 - t$ are (local) martingales. It follows that if $f \in C^2(\mathbf{R})$ then

$$f(M_t) = f(0) + \int_0^t f'(M_s)\, dM_s + \frac{1}{2}\int_0^t f''(M_s)\, ds, \tag{1}$$

where the stochastic integral $M_t^f := \int_0^t f'(M_s)\, dM_s$ is a local martingale. In particular, if f' is bounded then M_t^f is a martingale, in which case upon taking expectations in (1) we obtain

$$\mathbf{E}[f(M_t)] = f(0) + \frac{1}{2}\int_0^t \mathbf{E}[f''(M_s)]\, ds. \tag{2}$$

Let us take f in (2) to be of the form $f(x) = \exp(i\theta x)$, where $\theta \in \mathbf{R}$ and $i = \sqrt{-1}$. Writing $g(t) := \mathbf{E}[\exp(i\theta M_t)]$ we obtain

$$g(t) = 1 - \frac{\theta^2}{2}\int_0^t g(s)\, ds$$

because $f''(x) = -\theta^2 f(x)$. Consequently, g satisfies the initial value problem

$$g'(t) = -\frac{\theta^2}{2} g(t) \qquad g(0) = 1,$$

which has the unique solution $g(t) = \exp(-t\theta^2/2)$. Thus

$$\mathbf{E}[\exp(i\theta M_t)] = \exp(-t\theta^2/2), \qquad \theta \in \mathbf{R},$$

* *i.e.*, $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \ge 0$.

which means that $M_t \sim \mathcal{N}(0,t)$.

Now fix $s > 0$ and $A \in \mathcal{F}_s$ with $\mathbf{P}(A) > 0$. Define $\mathbf{P}^*(B) := \mathbf{P}(B \cap A)/\mathbf{P}(A) = \mathbf{P}(B|A)$, $\mathcal{F}_t^* := \mathcal{F}_{t+s}$, and $M_t^* := M_{t+s} - M_s$ for $t \geq 0$. Then with respect to the filtration $(\mathcal{F}_t^*)$ over the probability space $(\Omega, \mathcal{F}, \mathbf{P}^*)$, the stochastic process $(M_t^*)_{t\geq 0}$ is a continuous local martingale with $M_0^* = 0$ such that $[M_t^*]^2 - t$ is also a local martingale. The considerations of the preceding paragraph apply to this process, and we deduce that

$$\mathbf{E}^*[\exp(i\theta M_t^*)] = \exp(-t\theta^2/2). \tag{3}$$

Writing the "starred" objects explicitly, (3) becomes

$$\mathbf{E}[\exp(i\theta(M_{t+s} - M_t)); A] = \exp(-t\theta^2/2)\mathbf{P}(A). \tag{4}$$

Varying $A \in \mathcal{F}_s$ in (4) we find that

$$\mathbf{E}[\exp(i\theta(M_{t+s} - M_t))|\mathcal{F}_s] = \exp(-t\theta^2/2),$$

which shows that $M_{t+s} - M_s$ is independent of $\mathcal{F}_s$ and has the $\mathcal{N}(0,t)$ distribution. □

Example 1. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a filtered probability space, and let $W = (W_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$ Brownian motion. Let $H = (H_t)_{t\geq 0}$ be a measurable $(\mathcal{F}_t)$ adapted process taking on only the two values ± 1. Then $H \in \mathcal{L}^2$ so the stochastic integral

$$M_t := \int_0^t H_s \, dW_s, \qquad t \geq 0,$$

is a square-integrable martingale. Moreover, $\langle M \rangle_t = \int_0^t H_s^2 \, ds = \int_0^t 1 \, ds = t$, so $M_t^2 - t$ is also a local martingale. It follows from Lévy's theorem that M is also an $(\mathcal{F}_t)$ Brownian motion.

Example 2. As in the previous example, let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a filtered probability space, and let $W = (W_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$ Brownian motion. Now let $H = (H_t)_{t\geq 0}$ be an arbitrary element of $\mathcal{L}^2_{\text{loc}}$. As before we define the local martingale

$$M_t := \int_0^t H_s \, dW_s, \qquad t \geq 0,$$

which has quadratic variation process

$$\langle M \rangle_t = \int_0^t H_s^2 \, ds, \qquad t \geq 0.$$

Observe that $\langle M\rangle_t$ is continuous and non-decreasing. Let us assume that, almost surely,

$$\lim_{t\to\infty} \langle M\rangle_t = \infty. \tag{5}$$

Define

$$T(s) := \inf\{t : \langle M\rangle_t > s\}, \qquad s \geq 0.$$

Then (5) implies that $T(s)$ is finite (a.s.) for each $s \geq 0$, and it is not hard to check that each $T(s)$ is a stopping time. We use these stopping times to "time change" M into Brownian motion. Precisely, define

$$C_s := M_{T(s)}, \qquad s \geq 0,$$

and

$$\mathcal{G}_s := \mathcal{F}_{T(s)}, \qquad s \geq 0.$$

Then the stochastic process $C = (C_s)_{s\geq 0}$ is adapted to the filtration $(\mathcal{G}_s)_{s\geq 0}$. The optional stopping theorem implies that C is a local martingale (with respect to $(\mathcal{G}_s)$). Moreover, the quadratic variation interpretation of $\langle C\rangle$ and $\langle M\rangle$ implies that

$$\langle C\rangle_s = \langle M\rangle_{T(s)} = s, \qquad \forall s \geq 0,$$

almost surely. That is, $C_s^2 - s$ is also a $(\mathcal{G}_s)$ local martingale. Lévy's theorem now tells us that $C = (C_s)$ is a $(\mathcal{G}_s)$ Brownian motion. The moral: A continuous local martingale is just Brownian motion with its "clock" running too fast (or too slow).