Math 142A **Winter 2007**

Taylor's Theorem

As noted in class, the proof of Taylor's Theorem found in the text is slightly garbled. Below is what I hope is a more understandable version of that proof.

Theorem 5.4.8. (Taylor's Theorem) *Let $f : I \to \mathbf{R}$ where I is an open interval, and suppose that f has derivatives $f', f'', \ldots, f^{(n)}, f^{(n+1)}$ up to order $n+1$ at all points of I. Fix $a \in I$ and define*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \qquad x \in I.$$

For a given $x \in I$, the "error" $R_n(x) = f(x) - P_n(x)$ of approximating $f(x)$ by $P_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

where c is some point between x and a.

Proof. Fix a and x in I and let $P_n(x)$ be defined as above. Also, abbreviate $R_n(x)$ to R, bearing in mind that x is fixed. Define a new function F on I by the formula

$$F(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k + \frac{R}{(x-a)^{n+1}}(x-t)^{n+1}, \qquad t \in I.$$

(We are using t as the variable here because the customary letter x has already been taken.) Observe that

$$F(a) = f(x),$$

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and

$$F'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n - (n+1)\frac{R}{(x-a)^{n+1}}(x-t)^n.$$

This formula results from a telescoping effect that occurs upon differentiating the sum

$$\sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k;$$

indeed, the derivative of this sum is

$$f'(t) + [(x-t)f''(t) - f'(t)] + \left[\frac{(x-t)^2}{2}f'''(t) - f''(t)(x-t)\right] + \cdots$$
$$+ \left[\frac{(x-t)^n}{n!}f^{(n+1)}(t) - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right],$$

which collapses to yield

$$\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

We now apply Rolle's Theorem to F on the closed interval between x and a. The result is a number c strictly between x and a such that $F'(c) = 0$. This means that

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n - (n+1)\frac{R}{(x-a)^{n+1}}(x-c)^n = 0,$$

or

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n = (n+1)\frac{R}{(x-a)^{n+1}}(x-c)^n.$$

Cancelling $(x-c)^n$ from both sides we obtain

$$\frac{f^{(n+1)}(c)}{n!} = (n+1)\frac{R}{(x-a)^{n+1}},$$

and finally

$$R = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

in which c is some number between x and a. Recalling that $R = f(x) - P_n(x)$, we see that the theorem is proved. $\square$