

Math 280B, Winter 2008

Homework 9 — Due Wednesday, March 19

1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be sub- σ -fields of \mathcal{F} . We say that \mathcal{A} and \mathcal{C} are *conditionally independent* given \mathcal{B} provided

$$(CI) \quad \mathbf{P}[A \cap C | \mathcal{B}] = \mathbf{P}[A | \mathcal{B}] \cdot \mathbf{P}[C | \mathcal{B}], \quad \forall A \in \mathcal{A}, C \in \mathcal{C}.$$

Show that (CI) holds if and only if the following form of the *Markov Property* holds:

$$(MP) \quad \mathbf{P}[C | \mathcal{A} \vee \mathcal{B}] = \mathbf{P}[C | \mathcal{B}], \quad \forall C \in \mathcal{C}.$$

Remember that $\mathcal{A} \vee \mathcal{B} := \sigma(\mathcal{A} \cup \mathcal{B})$ and that (for example) $\mathbf{P}[A | \mathcal{B}] := \mathbf{E}[1_A | \mathcal{B}]$. [Hint: The collection of events of the form $A \cap B$ ($A \in \mathcal{A}, B \in \mathcal{B}$) is a π -system generating $\mathcal{A} \vee \mathcal{B}$.]

In problems 2–4, $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space with a filtration $(\mathcal{F}_n)_{n=0}^\infty$ such that $\mathcal{F}_n \subset \mathcal{F}$ for all n ; we define $\mathcal{F}_\infty := \bigvee_{n=0}^\infty \mathcal{F}_n \subset \mathcal{F}$.

2. Let ξ, ξ_1, ξ_2, \dots be elements of L^1 such that $\xi_n \xrightarrow{L^1} \xi$ as $n \rightarrow \infty$. Show that $\mathbf{E}[\xi_n | \mathcal{F}_n] \xrightarrow{L^1} \mathbf{E}[\xi | \mathcal{F}_\infty]$.

3. Let S and T be arbitrary stopping times. Show that $\mathbf{E}[\cdot | \mathcal{F}_S]$ and $\mathbf{E}[\cdot | \mathcal{F}_T]$ commute (as operators on L^1) and that their product is $\mathbf{E}[\cdot | \mathcal{F}_{S \wedge T}]$. That is, show that

$$\mathbf{E}[\mathbf{E}[X | \mathcal{F}_S] | \mathcal{F}_T] = \mathbf{E}[\mathbf{E}[X | \mathcal{F}_T] | \mathcal{F}_S] = \mathbf{E}[X | \mathcal{F}_{S \wedge T}], \quad \forall X \in L^1(\mathcal{F}).$$

4. Let $(B_n)_{n \geq 1}$ be a sequence of events with $B_n \in \mathcal{F}_n$ for each n . Define

$$G := \left\{ \omega \in \Omega : \sum_{n=1}^{\infty} 1_{B_n}(\omega) < \infty \right\},$$

and

$$H := \left\{ \omega \in \Omega : \sum_{n=1}^{\infty} \mathbf{P}[B_n | \mathcal{F}_{n-1}](\omega) < \infty \right\}.$$

Show that $\mathbf{P}[G \Delta H] = 0$; that is, for a.e. sample point $\omega \in \Omega$, the series $\sum_n 1_{B_n}(\omega)$ converges if and only if the series $\sum_n \mathbf{P}[B_n | \mathcal{F}_{n-1}](\omega)$ converges. [Hint: Recall from class the following result: If (X_n) is a martingale with $\mathbf{E}[\sup_n |\Delta X_n|] < \infty$, then $\mathbf{P}[C \cup D] = 1$, where

$$C = \{\lim_n X_n \text{ exists in } \mathbf{R}\}, \quad D = \{\limsup_n X_n = +\infty, \liminf_n X_n = -\infty\}.$$

Apply this result to the martingale

$$X_n := \sum_{k=1}^n \left\{ 1_{B_k} - \mathbf{P}[B_k | \mathcal{F}_{k-1}] \right\} \quad (X_0 = 0).$$

Notice that if the B_n s are mutually independent and $\mathcal{F}_n = \sigma\{B_1, \dots, B_n\}$, then the above assertion reduces to the Borel-Cantelli lemmas.]

5. Let $(\xi_n)_{n=1}^\infty$ be an iid sequence of integrable random variables, and let η be an integrable random variable, all defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Define $X_n := \xi_n + \eta$, $n = 1, 2, \dots$, and $\mathcal{F}_n := \sigma\{X_1, X_2, \dots, X_n\}$. Show that

$$\mathbf{E}[\eta | \mathcal{F}_n] \rightarrow \eta \text{ a.s. as } n \rightarrow \infty.$$