This exam is open book and open notes.

Show your work and justify your answers.

Each problem is worth 10 points.

---

1. Let X and Y be standard normal random variables, with correlation $\rho = .4$.

(a) The random variable $W = 2X + 3Y$ also has the normal distribution. Find $\mathbf{E}(W)$ and $\mathrm{Var}(W)$.

(b) For any real number b the random variable $V := X + bY$ has the normal distribution. For which value of b is V independent of X?

2. (a) Find the limiting distribution ($\pi_j = \lim_{n\to\infty} \Pr[X_n = j | X_0 = i]$) for the Markov chain $\{X_0, X_1, X_2, \ldots\}$ with state space $\{0, 1, 2\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/2 & 1/6 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

(b) Use the result of part (a) to find the mean return time $E(T_1 | X_0 = 1)$, where

$$T_1 = \min\{n \geq 1 : X_n = 1\}.$$

3. Determine the **communicating classes**, and the **period** of each state, for the Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}.$$

Which states are recurrent and which are transient?

4. Given a Poisson process $X(t)$, $t \geq 0$, of rate $\lambda > 0$, let us fix a time, say $t = 2$, and let us consider the first point of X to occur after time 2. Call this time W, so that $W = \min\{t > 2 : X(t) - X(2) = 1\}$. Show that the random variable $W - 2$ has the exponential distribution with parameter λ. [Hint: Begin by computing $\Pr[W - 2 > x]$ for $x \geq 0$.]

5. Be glad that you dwell in San Diego, and not in Cleveland. In Cleveland the weather on any given day can be described as either Fair, Bad, or Worse. Assume that the weather pattern in Cleveland evolves according to a Markov chain $X_0, X_1, \ldots$ with state space $\{F, B, W\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1/4 & 3/4 \\ 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \end{bmatrix}.$$

(a) $\Pr(X_2 = F | X_0 = i)$ for $i = F, B, W$.

(b) Find the "long run" probability of a fair day in Cleveland. That is, find

$$\lim_{n\to\infty} \Pr(X_n = F | X_0 = i)$$

where i is any of the states F, B, W.

6. Let $X(t)$ be the number of customers entering Volph's Supermarket during the first t minutes following 8 AM. (Thus $X(5) - X(2)$ is the number of customers arriving between 8:02 AM and 8:05 AM.) Assume that X is a Poisson process with rate $\lambda = 3$ customers arriving per minute.

(a) Calculate $\Pr[X(2) = 3, X(3) = 8]$.

(b) Let Z be the number of customers arriving between 9:15 AM and 9:30 AM. Express Z in terms of X.

(c) Compute $\mathbf{E}(Z)$.

7. Let $X(t)$, $t \geq 0$, be a Poisson process of rate $\lambda > 0$, and let $C_1, C_2, \ldots$ be independent and identically distributed Bernoulli random variables that are independent of X, with $\Pr[C_k = 1] = p$, $\Pr[C_k = 0] = q = 1 - p$. Consider the random variable

$$Y(t) = \sum_{k=1}^{X(t)} C_k,$$

where, as a matter of convention, $\sum_{k=1}^{0} = 0$. [Y is the "p-thinning" of X.]

(a) Explain why

$$\Pr[Y(1) = k | X(1) = n] = \binom{n}{k} p^k q^{n-k}, \qquad k = 0, 1, 2, \ldots, n.$$

(b) Compute $\Pr[Y(1) = 3, X(1) - Y(1) = 2]$.