Math 180B, Winter 2013

Homework 2

6.3.2. Let X and Y have the following joint density:

$$f(x,y) = \begin{cases} 2x + 2y - 4xy, & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal densities of X and Y.
(b) Find $f_Y(y|X = 1/4)$.
(c) Find $\mathbf{E}(Y|X = 1/4)$.

6.3.5. Suppose (X, Y) has uniform distribution on the triangle with vertices $(0, 1), (-1, 0), (1, 0)$. For x between -1 and 1, find
(a) $\mathbf{P}(Y \ge 1/2|X = x)$;
(b) $\mathbf{P}(Y < 1/2|X = x)$;
(c) $\mathbf{E}(Y|X = x)$;
(d) $\mathrm{Var}(Y|X = x)$.

6.3.9. Let A and B be events and let Y be a random variable uniformly distributed on $(0, 1)$. Suppose that, conditional on $Y = p$, A and B are independent, each with probability p. Find:
(a) the conditional probability of A given that B occurs;
(b) the conditional density of Y given that A occurs and B does not.

6.3.12. Suppose there are ten atoms, each of which decays by emission of an α-particle after an exponentially distributed lifetime with rate 1, independently of the others. Let T_1 be the time of the first α-particle emission, T_2 the time of the second. Find:
(a) the distribution of T_1;
(b) the conditional distribution of T_2 given T_1;
(c) the distribution of T_2.

6.4.2. Use the formula $\mathbf{P}(A) = \mathbf{P}(A|B)\mathbf{P}(B) + \mathbf{P}(A|B^c)\mathbf{P}(B^c)$ to prove:
(a) if $\mathbf{P}(A|B) = \mathbf{P}(A|B^c)$ then A and B are independent;
(b) if $\mathbf{P}(A|B) > \mathbf{P}(A|B^c)$ then A and B are positively dependent;
(c) if $\mathbf{P}(A|B) < \mathbf{P}(A|B^c)$ then A and B are negatively dependent;
Now prove the converses of (a), (b), and (c).

6.4.6. Let X_1 and X_2 be the numbers on two independent fair die rolls, $X = X_1 - X_2$ and $Y = X_1 + X_2$. Show that X and Y are uncorrelated but not independent.

6.4.8. You have N boxes labeled Box1, Box2,...,BoxN, and you have k balls. You drop the balls into the boxes, independently of each other. For each ball the probability that it will land in a particular box is the same for all boxes, namely $1/N$. Let X_1 be the number of balls in Box1, and X_N the number of balls in BoxN. Calculate $\mathrm{Corr}(X_1, X_N)$.

6.4.11. Let T_1 and T_3 be the times of the first and third arrivals in a Poisson process with rate λ. Find $\mathrm{Corr}(T_1, T_3)$.

6.4.21. A box contains 5 red balls and 8 blue ones. A random sample of size 3 is drawn *without* replacement. Let X be the number of red balls and let Y be the number of blue balls selected. Compute:
(a) $\mathbf{E}(X)$;
(b) $\mathbf{E}(Y)$;
(c) $\mathrm{Var}(X)$;
(d) $\mathrm{Corr}(X, Y)$.