Section 2.4, Exercise 1. We have \( p^* = (5/4 - 1/2)/(2 - 1/2) = 1/2. \)

(a) Evidently
\[
S_0 = 100, S_1(d) = 50, S_1(u) = 200,
\]
and
\[
V_1(d) = 0, V_1(u) = 100.
\]

Consequently,
\[
V_0 = \frac{1}{1 + 1/4} \cdot \frac{0 + 100}{2} = 40.
\]

(b) Using formulas (2.15) and (2.16) from the text we find that
\[
\alpha_1 = 2/3, \quad \beta_1 = -80/3.
\]

Check: \( \alpha_1 S_0 + \beta_1 B_0 = (2/3) \cdot 100 - (80/3) \cdot 1 = 120/3 = 40 = V_0, \) as desired.

(c) If \( C_0 = $41, \) then sell 1 call option, invest $40 in the hedging strategy of part (b), and put the remaining $1 in the bond. For this strategy (call it \( \tilde{\phi} \)) we have \( V_0(\tilde{\phi}) = 0 \) and \( V_1(\tilde{\phi}) = 5/4 > 0. \)

(d) We have the call/put parity formula
\[
C_0 - P_0 = S_0 - (1 + r)^{-1} K.
\]

Therefore the arbitrage-free price of a European put with strike $K and exercise time \( T = 1 \) is
\[
P_0 = C_0 - S_0 + (1 + r)^{-1} K = 40 - 100 + (4/5) \cdot 100 = 20.
\]

Section 2.4, Exercise 2. (a) Now \( p^* = .4. \) Evidently,
\[
S_0 = 100,
\]
\[
S_1(d) = 50, \quad S_1(u) = 200,
\]
\[
S_2(dd) = 25, \quad S_2(du) = S_2(ud) = 100, \quad S_2(uu) = 400.
\]

Because \( V_2 = X = (S_2 - 80)^+ \), we have
\[
V_2(dd) = 0, \quad V_2(du) = V_2(ud) = 20, \quad V_2(uu) = 320.
\]

Using the (undiscounted form of the) backward recursion (2.28) we find that
\[
V_1(d) = 80/11 = 7.27, \quad V_1(u) = 1400/11 = 127.27,
\]
and finally
\[
V_0 = 6080/121 = 50.25,
\]
The arbitrage-free price of this European call is therefore $50.25.

(b) The formulas (2.15) and (2.16) tells us

$$(\alpha_1, \beta_1) = (8/10, 600/121) = (0.8, -29.75)$$

and

$$(\alpha_2(d), \beta_2(d)) = (4/15, -2000/363) = (0.267, -5.51)$$
$$(\alpha_2(u), \beta_2(u)) = (1, -8000/121) = (1, -66.12).$$

This is the replicating strategy for the European call in question.

(c) Buy the option for $48.25; invest -$50.25 in $-\phi$ ($\phi$ is the replicating strategy from (b)); put the remaining $2 in the bond. For this strategy (call it $\tilde{\phi}$) $V_0(\tilde{\phi}) = 0$ while $V_1(\tilde{\phi}) = 2.42 > 0$.

Section 2.4, Exercise 4. In the expanded market allowing trade in stock, bond, and contingent claim, we first need the associated notion of trading strategy. This will be a sequence of triples

$$\psi = \{(\alpha_t, \beta_t, \gamma_t) : t = 1, 2, \ldots, T\},$$

such that (“predictability”)

$$\alpha_t, \beta_t, \gamma_t \text{ are each } \mathcal{F}_{t-1}\text{-measurable, for } t = 1, 2, \ldots, T,$$

and (\psi is “self-financing”)

$$V_t(\psi) := \alpha_t S_t + \beta_t B_t + \gamma_t C_s = \alpha_{t+1} S_t + \beta_{t+1} B_t + \gamma_{t+1} C_t, \quad t = 1, 2, \ldots, T - 1.$$

It is natural to assume (as we shall) that the time-$t$ price of the claim, namely the random variable $C_t$, is $\mathcal{F}_t$-measurable for $t = 0, 1, \ldots, T$.

An arbitrage opportunity is now represented by a trading strategy $\psi$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$, and $P[V_T(\psi) > 0] > 0$. Let us verify that the no-arbitrage time-$t$ price of the claim is $V_t(\phi^*)$, where $\phi^*$ is the hedging strategy (in stock and bond) for $X$.

Suppose first that for some $t \in \{0, 1, \ldots, T - 1\}$ the event $A := \{C_t > V_t(\phi^*)\}$ has positive probability. Consider the trading strategy $\psi$ defined by

$$(\alpha_s, \beta_s, \gamma_s) = (0, 0, 0), \quad s = 0, 1, \ldots, t - 1,$$

$$(\alpha_s, \beta_s, \gamma_s) = (0, 0, 0), \quad s = 0, 1, \ldots, T, \quad \text{if } A \text{ fails to occur},$$

$$(\alpha_s, \beta_s, \gamma_s) = (\alpha^*_s, \beta^*_s + (C_t - V_t(\phi^*))(1+r)^{-t}, -1), \quad s = t, \ldots, T, \quad \text{if } A \text{ occurs}.$$ 

This strategy can be described in words as follows. If $A$ does not occur, do nothing. If $A$ occurs, do nothing until time $t$; at that time short sell one claim (for $C_t$), buy into the replicating strategy $\phi^*$ (for $V_t(\phi^*)$), and invest the difference $C_t - V_t(\phi^*)$ in the bond. The cost (at time $t$) for this strategy is $0$ if $A^c$ occurs, while if $A$ occurs it is

$$-C_t + V_t(\phi^*) + (C_t - V_t(\phi^*)) = 0.$$
Thus, \( V_0(\psi) = V_1(\psi) = \cdots = V_t(\psi) = 0 \) in all cases. At time \( T \), the value of this portfolio is
\[
1_A \cdot (-X + V_T(\phi^*) + (C_t - V_t(\phi^*))(1 + r)^{T-t}) = 1_A \cdot (C_t - V_t(\phi^*))(1 + r)^{T-t},
\]
which is non-negative in all cases and strictly positive when \( A \) occurs. Thus \( \psi \) yields an arbitrage. This means that \( C_t > V_t(\phi^*) \) with positive probability for some \( t \) cannot happen.

In the same way, we cannot have \( C_t < V_t(\phi^*) \) with positive probability for some \( t \) without there being an arbitrage opportunity.

Finally, suppose that \( C_t = V_t(\phi^*) \) with probability 1, for all \( t \in \{0, 1, \ldots, T\} \). Suppose there is an arbitrage \( \psi \). I claim that the discounted value process \( V^*_t(\psi) \) is then a martingale. If this is the case then \( V^*_T(\psi) \) is a non-negative random variable with expectation
\[
E^*[V^*_T(\psi)] = E^*[V^*_0(\psi)] = V^*_0(\psi) = V_0(\psi) = 0,
\]
forcing \( V_T(\psi) \equiv 0 \), contradicting the assumption that \( \psi \) was an arbitrage.

To see the claim, compute (for \( t = 1, 2, \ldots, T \))
\[
E^*[V_t(\psi) | \mathcal{F}_{t-1}] = E^*[\alpha_t S_t + \beta_t B_t + \gamma_t C_t | \mathcal{F}_{t-1}]
\begin{align*}
&= \alpha_t E^*[S_t | \mathcal{F}_{t-1}] + \beta_t B_t + \gamma_t E^*[C_t | \mathcal{F}_{t-1}] \\
&= \alpha_t (1 + r) S_{t-1} + \beta_t B_t + \gamma_t (1 + r) C_{t-1} \\
&= (1 + r) [\alpha_t S_{t-1} + \beta_t B_{t-1} + \gamma_t C_{t-1}] \\
&= (1 + r) [\alpha_{t-1} S_{t-1} + \beta_{t-1} B_{t-1} + \gamma_{t-1} C_{t-1}] \\
&= (1 + r) V_{t-1}(\psi).
\end{align*}
\]
This means that \( V^*_t(\psi), t = 0, 1, 2, \ldots, T, \) is a martingale, as claimed. (In the second equality above we use the predictability of \( \alpha_t, \beta_t, \gamma_t \); the third equality uses the fact that \( S^*_t \) and \( C^*_t = V^*_t(\phi^*) \) are both martingales; the fifth uses the self-financing nature of \( \psi \).)