1. Show that $X_t = W_t^3 - 3tW_t$ is a martingale. [Hint: One way to do this is to compute the conditional expectation $E[X_t | F_s]$ ($0 < s < t$) by using the “trick” of writing $W_t = W_s + (W_t - W_s)$. Another is to use Itô’s formula to show that $X_t = 3 \int_0^t (W_s^2 - s) \, dW_s$.]

Solution 1. Let us fix $0 \leq s < t$. Because $W$ is a martingale, 

$$E[3tW_t | F_s] = 3tE[W_t | F_s] = 3tW_s.$$ 

To compute the conditional expectation of $W_t^3$ we use the indicated “trick” and the fact that the third moment of a mean-zero normal random variable is 0, by symmetry. Thus, abbreviating $Y := W_t - W_s$ (so that $Y$ is independent of $F_s$), 

$$E[W_t^3 | F_s] = E[(Y + W_s)^3 | F_s] = E[Y^3 + 3Y^2W_s + 3YW_s^2 + W_s^3 | F_s]$$ 


$$= 0 + 3W_s \cdot (t - s) + 3W_s^2 \cdot 0 + W_s^3$$ 

$$= 3(t - s)W_s + W_s^3.$$ 

It follows from these computations that 

$$E[X_t | F_s] = 3(t - s)W_s + W_s^3 - 3tW_s = W_s^3 - 3sW_s = X_s,$$ 

so $X$ is a martingale as claimed.

Solution 2. Let us apply Itô’s formula to the Brownian motion $W$ and the function $f(x,t) = x^3 - 3tx$. Observe that $f_x = 3x^2 - 3t$, $f_{xx} = 6x$, and $f_t = -3x$. Therefore, since $X_0 = 0$, 

$$X_t = f(W_t, t)$$ 

$$= \int_0^t (3W_s^2 - 3s) \, dW_s - \int_0^t 3W_s \, ds + \frac{1}{2} \int_0^t 6W_s \, ds$$ 

$$= \int_0^t (3W_s^2 - 3s) \, dW_s.$$ 

Being of the form $\int_0^t Z_s \, dW_s$ (where $Z_s := 3(W_s^2 - s)$ is an element of $L_{\text{loc}}$, as is easily verified), the process $X$ is necessarily a local martingale. But, because $(a + b)^2 \leq 2a^2 + 2b^2$, 

$$E[Z_s^2] = 9E[(W_s^2 - s)^2] \leq 18E[W_s^4] + 18s^2 = 72s^2,$$ 

so $\int_0^T Z_s^2 \, ds \leq 24T^3 < \infty$. Therefore $Z \in \mathcal{L}$, and the associated stochastic integral, namely the process $X$, is a true martingale.
2. Fix $\sigma > 0$ and $\mu \in \mathbb{R}$. Express $X_t := \exp(\sigma W_t + \mu t)$ as an Itô process; that is, find $Y$ and $b$ such that $X_t = X_0 + \int_0^t Y_s dW_s + \int_0^t b_s ds$. Under what condition on $\sigma$ and $\mu$ is $X$ a martingale?

Solution. We apply Itô’s formula to $f(x, t) = \exp(\sigma x + \mu t)$, for which $f_x = \sigma f$, $f_{xx} = \sigma^2 f$, and $f_t = \mu f$. The result is

$$X_t = 1 + \sigma \int_0^t X_s dW_s + \mu \int_0^t X_s ds + \frac{\sigma^2}{2} \int_0^t X_s ds$$

Thus $X$ is an Itô process with

$$Y_s = \sigma X_s, \quad b_s = [\mu + (\sigma^2/2)] X_s.$$  

Clearly $X$ is a local martingale if and only if $b$ vanishes; that is, if and only if $\mu + \sigma^2/2 = 0$. In this case, because

$$\mathbb{E}[X_s^2] = \mathbb{E}[\exp(2\sigma W_s + 2\mu s)] = \exp(2\sigma^2 s + 2\mu s),$$

we have

$$\int_0^T \mathbb{E}[Y_s^2] ds = \sigma^2 \int_0^T \exp(2(\sigma^2 + \mu) s) ds = \frac{\sigma^2}{2(\sigma^2 + \mu)} \left(e^{2(\sigma^2 + \mu) T} - 1\right) < \infty,$$

and so $X$ is, in fact, a martingale.


Solution. If $\phi$ were an admissible strategy, then $(V_t(\phi))_{0 \leq t \leq T}$ would be a martingale, and we would have $\mathbb{E}[V_T(\phi) | \mathcal{F}_t] = V_t(\phi)$ for each fixed $t \in (0, T)$. But $V_T(\phi) = I(T \land \tau_a) = I(\tau_a) = a$ is non-random, whereas $V_t(\phi) = I(t)$ on the event $\{t < \tau_a\}$. Notice that $I(t)$ is normally distributed with mean 0 and variance $\log(T/(T - t)) > 0$. Thus, if $\phi$ were admissible we would have $I(t) = a$ with positive probability, which is impossible for a normally distributed random variable. Thus $\phi$ is not admissible.


Solution. (a) According to formula (4.62), the price of a European call option in the Black-Scholes setting is

$$C_0 = S_0 \Phi \left(\frac{\log(S_0/K^*)}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2}\right) - K^* \Phi \left(\frac{\log(S_0/K^*)}{\sigma \sqrt{T}} - \frac{\sigma \sqrt{T}}{2}\right)$$

$$= 10 \cdot \Phi \left(\frac{\log(10/11.7037)}{.2 \cdot .7071} + \frac{.2 \cdot .7071}{2}\right) - 11.7037 \cdot \Phi \left(\frac{\log(10/11.7037)}{.2 \cdot .7071} - \frac{.2 \cdot .7071}{2}\right)$$

$$= 10 \cdot \Phi (-1.04) - 11.7037 \cdot \Phi (-1.18)$$

$$= 10 \cdot [1 - \Phi (1.04)] - 11.7037 \cdot [1 - \Phi (1.18)]$$

$$= 10 \cdot [.1492] - 11.7037 \cdot [.1190]$$

$$= .099.$$
which rounds off to 10¢. A replicating strategy \( \phi = (\alpha_t, \beta_t)_{0 \leq t \leq .5} \) is given by the formulae on page 74 of the text, in the lines displayed below (4.69).

(b) Our payoff in this case is \( X = 1_{\{S_{.5} > 10\}} \), which has time-zero no-arbitrage price

\[
C_0 = \mathbb{E}^*[X^*] = \mathbb{E}^*[1_{\{S_{.5} > 10\}}] e^{-rT} = \mathbb{P}^*[S_{.5} > 10] e^{-0.025}.
\]

But \( S_t = S_0 \exp(rt + \sigma\tilde{W}_t - \sigma^2t/2) \) and under \( \mathbb{P}^* \), \( \tilde{W}_t \) is normally distributed with mean 0 and variance \( t \). Therefore

\[
\mathbb{P}^*[S_t > 10] = \mathbb{P}^*[\tilde{W}_t > (\log(10/S_0) - rt + \sigma^2t/2)/\sigma] = 1 - \Phi \left( \frac{\log(10/S_0) - rt + \sigma^2t/2}{\sigma\sqrt{t}} \right)
\]

Inserting the numerical values for the various parameters we find that

\[
\mathbb{P}^*[S_{.5} > 10] = 1 - \Phi \left( \frac{\log(10/10) - .025 + .01}{.2 \cdot \sqrt{1/2}} \right)
= 1 - \Phi \left( -.015 \cdot 5\sqrt{2} \right)
= 1 - \Phi (.1061) = \Phi (.1061) = .542.
\]

Thus, \( C_0 = .57 \cdot e^{-0.025} = .529 \).