In problems 1 and 2, $W = (W_t)_{0 \leq t \leq T}$ is a (standard) Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, and $\mathcal{F} = \mathcal{F}_T$.

1. Let $\mu$ and $\nu$ be real numbers. With $W$ as described above, the process $X_t := W_t + \mu t$ is a Brownian motion with drift $\mu$. Using Girsanov’s theorem, find an equivalent probability measure $Q$ (on $(\Omega, \mathcal{F}_T)$) under which $(X_t)_{0 \leq t \leq T}$ is a Brownian motion with drift $\nu$. That is, $X_t = \tilde{W}_t + \nu t$, $0 \leq t \leq T$, where $\tilde{W}_t := W_t + (\mu - \nu)t$ is a $Q$-Brownian motion.

Solution. To make $W_t + (\mu - \nu)t$ into a martingale, we use Girsanov’s theorem with the density

$$\Lambda_T := \exp \left( (\nu - \mu)W_T - (\nu - \mu)^2 t / 2 \right).$$

The probability measure $Q$ defined on $\mathcal{F}_T$ by

$$Q(A) := \mathbb{P}[1_A \Lambda_T], \quad A \in \mathcal{F}_T,$$

is equivalent to $\mathbb{P}$ and $\tilde{W}_t := W_t - (\nu - \mu)t = W_t + (\mu - \nu)t$ is a $Q$-Brownian motion. Consequently, $X_t = W_t + \mu t = W_t + (\mu - \nu)t + \nu t = \tilde{W}_t + \nu t$ is a Brownian motion with drift $\nu$ under $Q$.

2. Consider, in the Black-Scholes setup, a European contingent claim (called a forward start option) whose payoff at time $T$ is

$$X = (S_T - S_{T_0})^+,$$

where $T_0 \in (0, T)$ is a fixed time. This is like a European call option, except that the strike price is (the random variable) $S_{T_0}$. Determine the no-arbitrage time-zero price of this option $X$. [Hint: Recall that

$$S^*_T = S^*_{T_0} \exp \left( \sigma (\tilde{W}_T - \tilde{W}_{T_0}) - \sigma^2 (T - T_0) / 2 \right)$$

and that $\tilde{W}_T - \tilde{W}_{T_0}$ is independent of $\mathcal{F}_{T_0}$ (in particular, independent of $S^*_{T_0}$). Thus

$$X^* = \left( S^*_T - S^*_{T_0} e^{-r(T - T_0)} \right)^+ = S^*_{T_0} \cdot \left( \tilde{S}^*_{T - T_0} - e^{-r(T - T_0)} \right)^+,$$
where \( \hat{S}_{T-T_0}^* = \exp \left( \sigma (\hat{W}_T - \hat{W}_{T_0}) - \sigma^2 (T - T_0)/2 \right) \).

**Solution.** Using the hint, we have

\[
C_0 = \mathbb{E}^*[X^*] = \mathbb{E}^*[S_{T_0}^*] \cdot \mathbb{E}^* \left[ \left( \hat{S}_{T-T_0}^* - e^{-r(T-T_0)} \right)^+ \right]
\]

\[
= S_0 \cdot \mathbb{E}^* \left[ \left( \hat{S}_{T-T_0}^* - e^{-r(T-T_0)} \right)^+ \right],
\]

because \( (S_t^*)_{t \geq 0} \) is a \( \mathbb{P}^* \)-martingale. The second factor on the extreme right side of (2.1) is the price of a European call option with strike price \( e^{-r(T-T_0)} \) and terminal time \( T - T_0 \), when the stock has initial price 1, and the drift and interest rate are taken to vanish. By the Black-Scholes formula, that price is

\[
\Phi \left( \frac{\log(1/e^{-r(T-T_0)})}{\sigma \sqrt{T-T_0}} + \frac{\sigma \sqrt{T-T_0}}{2} \right) - e^{-r(T-T_0)} \Phi \left( \frac{\log(1/e^{-r(T-T_0)})}{\sigma \sqrt{T-T_0}} - \frac{\sigma \sqrt{T-T_0}}{2} \right),
\]

which simplifies to

\[
\Phi \left( \sqrt{T-T_0} \left( \frac{r}{\sigma} + \frac{\sigma}{2} \right) \right) - e^{-r(T-T_0)} \Phi \left( \sqrt{T-T_0} \left( \frac{r}{\sigma} - \frac{\sigma}{2} \right) \right).
\]

The price of our option is therefore

\[
C_0 = S_0 \cdot \left[ \Phi \left( \sqrt{T-T_0} \left( \frac{r}{\sigma} + \frac{\sigma}{2} \right) \right) - e^{-r(T-T_0)} \Phi \left( \sqrt{T-T_0} \left( \frac{r}{\sigma} - \frac{\sigma}{2} \right) \right) \right].
\]

**3. Section 4.10, Exercise 3.**

**Solution.** Formula (4.62) gives

\[
C_0 = S_0 \Phi \left( \frac{\log(S_0/K^*)}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2} \right) - K^* \Phi \left( \frac{\log(S_0/K^*)}{\sigma \sqrt{T}} - \frac{\sigma \sqrt{T}}{2} \right).
\]

To verify that \( C_0 \) is a strictly increasing function of \( \sigma \), we compute the derivative of \( C_0 \) with respect to \( \sigma \). To shorten the formulas we abbreviate \( \alpha := \log(S_0/K^*)/\sqrt{T} \) and \( \beta := \sqrt{T}/2 \).

With this notation we have \( C_0 = S_0 \Phi(\alpha/\sigma + \beta \sigma) - K^* \Phi(\alpha/\sigma - \beta \sigma) \), so

\[
\frac{\partial C_0}{\partial \sigma} = S_0 \varphi(\alpha/\sigma + \beta \sigma) \cdot (\beta - \alpha/\sigma^2) + K^* \varphi(\alpha/\sigma - \beta \sigma) \cdot (\beta + \alpha/\sigma^2).
\]

But

\[
\varphi(\alpha/\sigma - \beta \sigma) = e^{2\alpha \beta} \varphi(\alpha/\sigma + \beta \sigma)
\]

and \( e^{2\alpha \beta} = S_0/K^* \), so

\[
\frac{\partial C_0}{\partial \sigma} = \varphi(\alpha/\sigma + \beta \sigma) \left[ S_0(\beta - \alpha/\sigma^2) + K^* e^{2\alpha \beta}(\beta + \alpha/\sigma^2) \right]
\]

\[
= \varphi(\alpha/\sigma + \beta \sigma) \left[ S_0(\beta - \alpha/\sigma^2) + S_0(\beta + \alpha/\sigma^2) \right]
\]

\[
= \varphi(\alpha/\sigma + \beta \sigma) S_0 \cdot 2\beta > 0.
\]

It follows that \( C_0 \) is a strictly increasing function of \( \sigma > 0 \).

Solution. We are considering a European call option in the Black-Scholes context, with parameter values equal to $K = 70$, $T = 1/12$ (years), $r = 0.06$, $S_0 = 82$, and with $C_0$ given to be $14$. The “implied volatility” $\sigma$ is therefore given implicitly by

\begin{equation}
14 = 82\Phi\left(\frac{\log(82/(70e^{-0.005})) + \sigma}{\sigma\sqrt{1/12}}\right) - \frac{70}{e^{0.005}}\Phi\left(\frac{\log(82/(70e^{-0.005})) - \sigma}{\sigma\sqrt{1/12}}\right)
\end{equation}

(4.1)

\begin{equation}
= 82\Phi\left(\frac{0.5654}{\sigma} + 0.1443\sigma\right) - 69.6509\Phi\left(\frac{0.5654}{\sigma} - 0.1443\sigma\right).
\end{equation}

In view of Exercise 4.10.3, this equation has a unique solution, because the extreme right side of (4.1) has limit $82 - 67.9312 = 12.3491 < 14$ when $\sigma \to 0^+$ and limit $82 > 14$ when $\sigma \to +\infty$. The equation can be solved numerically in various ways (Newton’s method, bisection method,...). Mathematica’s FindRoot command yields the solution 0.674987.