1. [20 points]. The random variable \( Y \) is uniformly distributed on the interval \((0, 1)\) and the conditional distribution of the random variable \( X \), given that \( Y = y \), is uniform on the interval \((0, y)\).

(a) Find \( E[X] \). [Hint: Law of the Forgetful Statistician.]

(b) Find the joint density \( f(x, y) \) of \( X \) and \( Y \).

(c) Find the marginal density \( f_X(x) \) of \( X \).

Solution. (a) Because of the given conditional distribution of \( X \), we have \( E[X|Y = y] = y/2 \) for each \( y \in (0, 1) \). Therefore \( E[X|Y] = Y/2 \). Thus, by the Law of the Forgetful Statistician

\[ E[X] = E[E[X|Y]] = E[Y/2] = \frac{1}{2} E[Y] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}. \]

(b) We are given that

\[ f_{X|Y}(x|y) = \begin{cases} 1/y, & 0 < x < y < 1, \\ 0, & \text{otherwise} \end{cases} \]

and

\[ f_Y(y) = \begin{cases} 1, & 0 < y < 1, \\ 0, & \text{otherwise} \end{cases} \]

Therefore

\[ f(x, y) = f_{X|Y}(x|y) \cdot f_Y(y) = f_{X|Y}(x|y) = \begin{cases} 1/y, & 0 < x < y < 1, \\ 0, & \text{otherwise}. \end{cases} \]

(c) Integrating we obtain

\[ f_X(x) = \int_{-\infty}^{\infty} f(x, y), dy = \int_{x}^{1} \frac{1}{y} dy = \log y \bigg|_{x}^{1} = -\log x = \log(1/x), \quad 0 < x < 1. \]

[This formula can be used to give a more difficult derivation of the result found in part (a):

\[ E[X] = \int_{0}^{1} xf_X(x) dx = \int_{0}^{1} x \log(1/x) \\
= \frac{x^2}{2} \log(1/x) \bigg|_{0}^{1} - \int_{0}^{1} \frac{x^2}{2} \cdot -\frac{1}{x} dx \\
= 0 - 0 + \int_{0}^{1} \frac{x}{2} dx = \frac{1}{4}. \]

2. [20 points]. The random variables \( X \) and \( Y \) have a bivariate normal distribution with \( E[X] = E[Y] = 0 \), \( \text{Var}(X) = \text{Var}(Y) = 1 \), and correlation \( \rho = 1/2 \). Define new random variables \( Z \) and \( W \) by \( Z = X + Y \) and \( W = X - Y \).

(a) Find \( \text{Var}[Z] \) and \( \text{Var}[W] \).

(b) Find \( \text{Cov}[Z, W] \).
(c) Find $E[Z|W = w]$. [Hint: $Z$ and $W$ have a bivariate normal distribution. (Why?) Use this and the result of part (b).]

Solution. (a)

$$\text{Var}[Z] = \text{Var}[X] + \text{Var}[Y] + 2 \text{Cov}[X,Y] = 1 + 1 + 2 \cdot \frac{1}{2} = 3.$$  
Likewise  

$$\text{Var}[W] = \text{Var}[X] + \text{Var}[Y] - 2 \text{Cov}[X,Y] = 1 + 1 - 2 \cdot \frac{1}{2} = 1.$$  

(b) Because both $X$ and $Y$ have mean 0, so do $Z$ and $W$. Therefore


$$= E[X^2 - Y^2] = E[X^2] - E[Y^2]$$

$$= \text{Var}[X] - \text{Var}[Y] = 1 - 1 = 0.$$  

(c) As hinted, $Z$ and $W$ have a bivariate normal distribution. This is because

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix},$$

so that $\begin{bmatrix} Z \\ W \end{bmatrix}$ is a linear transformation of $\begin{bmatrix} X \\ Y \end{bmatrix}$. By part (b), $Z$ and $W$ are uncorrelated, hence independent! Therefore $E[Z|W = w] = E[Z] = 0.$

3. [20 points]. The Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ has state space $\{0, 1, 2\}$, transition matrix

$$P = \begin{bmatrix} 0 & .6 & .4 \\ .5 & .2 & .3 \\ .5 & 0 & .5 \end{bmatrix},$$

and initial distribution $p_i = P[X_0 = i]$ given by

$$[p_0 \ p_1 \ p_2] = [1/5 \ 2/5 \ 2/5].$$

(a) Find $P[X_1 = 2]$.

(b) Find $P[X_2 = 1|X_0 = 2]$.

(c) Find $P[X_{21} = 1, X_{22} = 2|X_{19} = 2]$.

Solution. (a)

$$P[X_1 = 2] = \sum_{i=0}^{2} P[X_0 = i]P[X_1 = 2|X_0 = i]$$

$$= \sum_{i=0}^{2} p_i \cdot P_{i,2}$$

$$= .2 \cdot .4 + .4 \cdot .3 + .4 \cdot .5$$

$$= .4$$
(b) By matrix multiplication,

\[ P^2 = \begin{bmatrix} .50 & .12 & .38 \\ .25 & .34 & .41 \\ .25 & .30 & .45 \end{bmatrix}. \]

In particular,

\[ P[X_2 = 1|X_0 = 2] = P^{(2)}_{2,1} = [P^2]_{2,1} = .3. \]

[Of course, you don’t need to compute \( P^2 \) completely; all you really need is the entry in row 2 and column 1.]

(c) Using the Markov property for the first equality below, we compute:

\[
P[X_{21} = 1, X_{22} = 2|X_{19} = 2] = P[X_{21} = 1|X_{19} = 2] \cdot P[X_{22} = 2|X_{21} = 1]
= P^{(2)}_{2,1} \cdot P_{1,2}
= .3 \cdot .3 = .9.
\]

4. [20 points]. A Markov chain \( \{X_n : n = 0, 1, 2, \ldots \} \) has state space \( \{0, 1, 2\} \) and transition matrix

\[
P = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}.
\]

(a) Find the stationary distribution \( \pi = (\pi_0, \pi_1, \pi_2) \) for this chain.

(b) Use the result of part (a) to find the mean return time \( E(T_1|X_0 = 1) \), where \( T_1 \) is the first time \( n \geq 1 \) such that \( X_n = 1 \).

Solution. (a) Because \( P \) is doubly stochastic (cf. Problem 5), the uniform distribution

\[
\pi = [1/3 \ 1/3 \ 1/3]
\]

satisfies \( \pi P = \pi \). This \( \pi \) is the unique stationary distribution because \( P \) is clearly irreducible (even regular: all entries of \( P^2 \) are strictly positive).

(b) By a general result discussed in class, the mean return time \( m_{1,1} = E[T_1|X_0 = 1] \) is reciprocal to \( \pi_1 \). Therefore, by part (a), \( m_{1,1} = 3. \)

5. [20 points]. Let \( P \) be the transition matrix of a Markov chain with 10 states. Assume that \( P \) is regular and doubly stochastic.

(a) Explain the meaning of the terms “regular” and “doubly stochastic.”

(b) Explain why the \( n \)-step transition probabilities satisfy

\[
\lim_{n \to \infty} P^{(n)}_{ij} = 1/10, \quad \text{for all } i, j.
\]

Solution. Assume the state space is \( \{0, 1, \ldots, 9\} \).

(a) A transition matrix \( P \) is regular provided there is a positive integer \( n \) such that \( P^{(n)}_{ij} > 0 \) for all states \( i \) and \( j \); that is, the \( n \)-th power \( P^n \) has only strictly positive entries. A stochastic matrix \( P \) is doubly stochastic provided the column sums of \( P \) are all 1 (and not just the row sums).
Because $P$ is regular it admits a unique stationary distribution, and then because $P$ is doubly stochastic the uniform distribution on $\{0,1,\ldots,9\}$, call it $\pi$, satisfies $\pi P = \pi$. Thus the uniform distribution $\pi_i = 1/10$ for $i = 0,1,\ldots,9$ is the stationary distribution. Because $P$ is regular, the basic limit theorem holds, and the limits of the transition probabilities coincide with the stationary distribution; that is,

$$\lim_{n\to\infty} P^{(n)}_{i,j} = \pi_j = 1/10, \quad \forall i,j.$$ 

6. [20 points]. The transition matrix for a Markov chain on the state space $\{0,1,2,3\}$ is

$$P = \begin{bmatrix}
0 & 1/2 & 0 & 1/2 \\
1/2 & 0 & 1/2 & 0 \\
0 & 1/2 & 0 & 1/2 \\
1/2 & 0 & 1/2 & 0
\end{bmatrix}.$$ 

(Think of a bug performing a random walk on a necklace with 4 beads.) Assuming that the chain starts in state 0, what is the probability that it gets to state 2 before state 3? [Hint: Alter the Markov chain to make states 2 and 3 absorbing, and then find the probability of being absorbed in state 2 if the chain starts in state 0.]

**Solution.** Let us alter $P$ to make 2 and 3 absorbing states. The resulting transition matrix is

$$Q = \begin{bmatrix}
0 & 1/2 \\
1/2 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix} = R.$$ 

We partition the state space as $\{0,1\}$ and $\{2,3\}$ and the transition matrix likewise, and then use the matrix method to compute absorption probabilities. The associated matrices are

$$Q = \begin{bmatrix}
0 & 1/2 \\
1/2 & 0
\end{bmatrix} = R.$$ 

We have

$$I - Q = \begin{bmatrix}
1 & -1/2 \\
-1/2 & 1
\end{bmatrix},$$

which has determinant $3/4$. Therefore

$$(I - Q)^{-1} = \frac{1}{4/3} \begin{bmatrix}
1 & 1/2 \\
1/2 & 1
\end{bmatrix} = \begin{bmatrix}
4/3 & 2/3 \\
2/3 & 4/3
\end{bmatrix}.$$ 

Writing $T = \min\{n : X_n \in \{2,3\}\}$ for the absorption time, the absorption probabilities $v_{ij} = P[X_T = j|X_0 = i]$ for $i = 0,1$ and $j = 2,3$ are the entries of the matrix

$$V = (I - Q)^{-1} R = \begin{bmatrix}
4/3 & 2/3 \\
2/3 & 4/3
\end{bmatrix} \begin{bmatrix}
0 & 1/2 \\
1/2 & 0
\end{bmatrix} = \begin{bmatrix}
1/3 & 2/3 \\
2/3 & 1/3
\end{bmatrix}.$$ 

In particular,

$$P[T_2 < T_3|X_0 = 0] = P[X_T = 2|X_0 = 0] = v_{02} = 1/3.$$
7. [20 points]. Customers arrive at a service counter in accordance with a Poisson process of rate $7$ (customers per hour). Assume that each customer requires exactly 2 hour of service, and that the number of servers is unlimited. Fix $t > 2$.

(a) Find the probability that there are no customers being served at time $t$.

(b) Find the expected number of customers being served at time $t$.

Solution. (a) There are no customers being served at time $t$ if and only if no customers arrived in the time interval $[t - 2, t]$. But the number of customers that arrive in $[t - 2, t]$ has the Poisson distribution with parameter $7 \cdot 2 = 14$. The probability that this count is 0 is therefore $e^{-14} = 8.32 \times 10^{-7}$.

(b) As noted already in part (a), the number of customers being served at time $t$ is $N[t - 2, t]$, the number of arrivals in $[t - 2, t]$, and this random variable has the Poisson distribution with parameter 14. The expected number of customers in service at time $t$ is therefore 14.

8. [20 points]. Let $\{X(t) : t \geq 0\}$ be a Poisson process of rate $\lambda > 0$, and let $0 < s < t$ be two times. Compute the following:

(a) $P[X(s) = 3, X(t) = 5]$.

(b) $E[X(t)|X(s) = 3]$.

(c) $\text{Corr}(X(s), X(t))$.

Solution. (a) Because $X(s)$ and $X(t) - X(s)$ are independent,

$$P[X(s) = 3, X(t) = 5] = P[X(s) = 3, X(t) - X(s) = 2]$$

$$= P[X(s) = 3] \cdot P[X(t) - X(s) = 2]$$

$$= e^{-\lambda s} \left( \frac{\lambda s}{3!} \right)^3 \cdot e^{-\lambda (t-s)} \left( \frac{\lambda (t-s)}{2!} \right)^2$$

$$= e^{-\lambda t} \frac{\lambda^5 s^3 (t-s)^2}{12}.$$ 

(b) As for part (a), $X(s)$ and $X(t) - X(s)$ are independent, and so

$$E[X(t) - X(s)|X(s) = 3] = E[X(t) - X(s)] = \lambda (t-s).$$

Therefore,

$$E[X(t)|X(s) = 3] = E[X(s)|X(s) = 3] + E[X(t) - X(s)|X(s) = 3] = 3 + \lambda (t-s).$$

(c) This correlation was computed in class, and shown there to be equal to $\lambda s$. This can be verified by using the independence noted in (a) and (b): Because $X(s)$ and $X(t) - X(s)$ are independent, their covariance is 0. Therefore

$$\text{Cov}[X(s), X(t)] = \text{Cov}[X(s), X(s)] + \text{Cov}[X(s), X(t) - X(t)]$$

$$= \text{Var}[X(s)] + 0 = \lambda s.$$ 

The third equality follows because $X(s)$ has the Poisson distribution with parameter $\lambda s$. 

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9. [20 points]. Let \( W_1, W_2, \ldots \) be the arrival times in a Poisson process of rate \( \lambda > 0 \), and let \( X(t) = N(0,t] \) be the number of arrivals in the interval \((0,t]\).

(a) Find the conditional mean \( E[W_1|X(t) = 2] \).

(b) Find the conditional mean \( E[W_3|X(t) = 5] \).

(c) Compute
\[
E \left[ \sum_{k=1}^{X(t)} W_k^2 \right].
\]

(By convention, \( \sum_{k=1}^{0} W_k^2 = 0 \).)

Solution. (a) From class discussion we know that the conditional distribution of \( W_1 \), given that \( X(t) = 2 \), is the order statistic \( U(1) \) from a sample of size 2 from the uniform distribution on \((0,t]\). Because the spacings \( U(1), U(2) - U(1), \ldots, U(n) - U(n-1), t - U(n) \) (of which there are \( n + 1 \) in a sample of size \( n \)) all have the same distribution and add up to \( t \), their individual mean values must be \( t/(n+1) \). In particular (when \( n = 2 \)), \( E[U(1)] = t/3 \). Therefore,
\[
E[W_1|X(t) = 2] = t/3.
\]

(b) Using the spacings fact cited in part (a),
\[
E[W_3|X(t) = 5] = E[U(3)],
\]
where \( U(3) \) is the third order statistic from a uniform sample of size 5, so that
\[
E[W_3|X(t) = 5] = E[U(3) - U(2)] + E[U(2) - U(1)] + E[U(1)] = 3 \cdot \frac{t}{6} = t/2.
\]

(c) We use the Law of Total Probability. First, by the Poisson – Uniform Order Statistics theorem and the symmetry of the summation process,
\[
E \left[ \sum_{k=1}^{X(t)} W_k^2 \right] = E \left[ \sum_{k=1}^{n} U_{(k)}^2 \right] = E \left[ \sum_{k=1}^{n} U_k^2 \right] = n \cdot E[U_1^2].
\]
Also, because \( U_1 \) has the uniform distribution on \((0,t]\), \( E[U_1^2] = t^{-1} \int_0^t u^2 \, du = t^2/3 \). Finally,
\[
E \left[ \sum_{k=1}^{X(t)} W_k^2 \right] = E \left[ \sum_{k=1}^{X(t)} W_k^2 | X(t) = n \right] \cdot P[X(t) = n]
\]
\[
= \sum_{n=1}^{\infty} \frac{nt^2}{3} \cdot \lambda \cdot \frac{\lambda t^3}{3}.
\]