1. Use the formula \( P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) \) to prove that if \( P(A|B) = P(A|B^c) \) then \( A \) and \( B \) are independent. Then prove the converse (that if \( A \) and \( B \) are independent then \( P(A|B) = P(A|B^c) \)). [Assume that \( P(B) > 0 \) and \( P(B^c) > 0 \).]

2. Let \( X_1 \) and \( X_2 \) be the numbers showing when two fair dice are thrown. Define new random variables \( X = X_1 - X_2 \) and \( Y = X_1 + X_2 \). Show that \( X \) and \( Y \) are uncorrelated but not independent. [Hint: To show lack of independence, it is enough to show that \( P[X = j, Y = k] \neq P[X = j] \cdot P[Y = k] \) for one pair \((j, k)\); try the pair \((0, 2)\).]

3. You have \( N \) boxes (labeled \( 1, 2, \ldots, N \)), and you have \( k \) balls. You drop the balls into the boxes, independently of each other. For each ball the probability that it will land in a particular box is \( 1/N \). Let \( X_1 \) be the number of balls in box \( 1 \) and \( X_N \) the number of balls in box \( N \). Calculate \( \text{Corr}(X_1, X_N) \).

4. Suppose \( X \) and \( Y \) are standard normal random variables. Find an expression for \( P(X + 2Y \leq 3) \) in terms of the standard normal distribution function \( \Phi \) in two cases:
   (a) \( X \) and \( Y \) are independent;
   (b) \( X \) and \( Y \) have bivariate normal distribution with correlation \( \rho = 1/2 \).

5. Let \( X_1 \) and \( X_2 \) be two independent standard normal random variables. Define two new random variables as follows: \( Y_1 = X_1 + X_2 \) and \( Y_2 = X_1 + \beta X_2 \). You are not given the constant \( \beta \) but it is known that \( \text{Cov}(Y_1, Y_2) = 0 \). Find
   (a) the density of \( Y_2 \);
   (b) \( \text{Cov}(X_2, Y_2) \).

6. Suppose that \((W, Z)\) have a bivariate normal distribution, that \( W \sim \mathcal{N}(0, 1) \), and that the conditional distribution of \( Z \), given that \( W = w \), is \( \mathcal{N}(aw + b, \tau^2) \).
   (a) What is the marginal distribution of \( Z \)?
   (b) What is the conditional distribution of \( W \), given that \( Z = z \)?

In addition:

**Pages 64–65**: Exercises 2.3.1, 2.3.5; Problems 2.3.2, 2.3.4(a)

**Pages 70-71**: Exercise 2.4.3